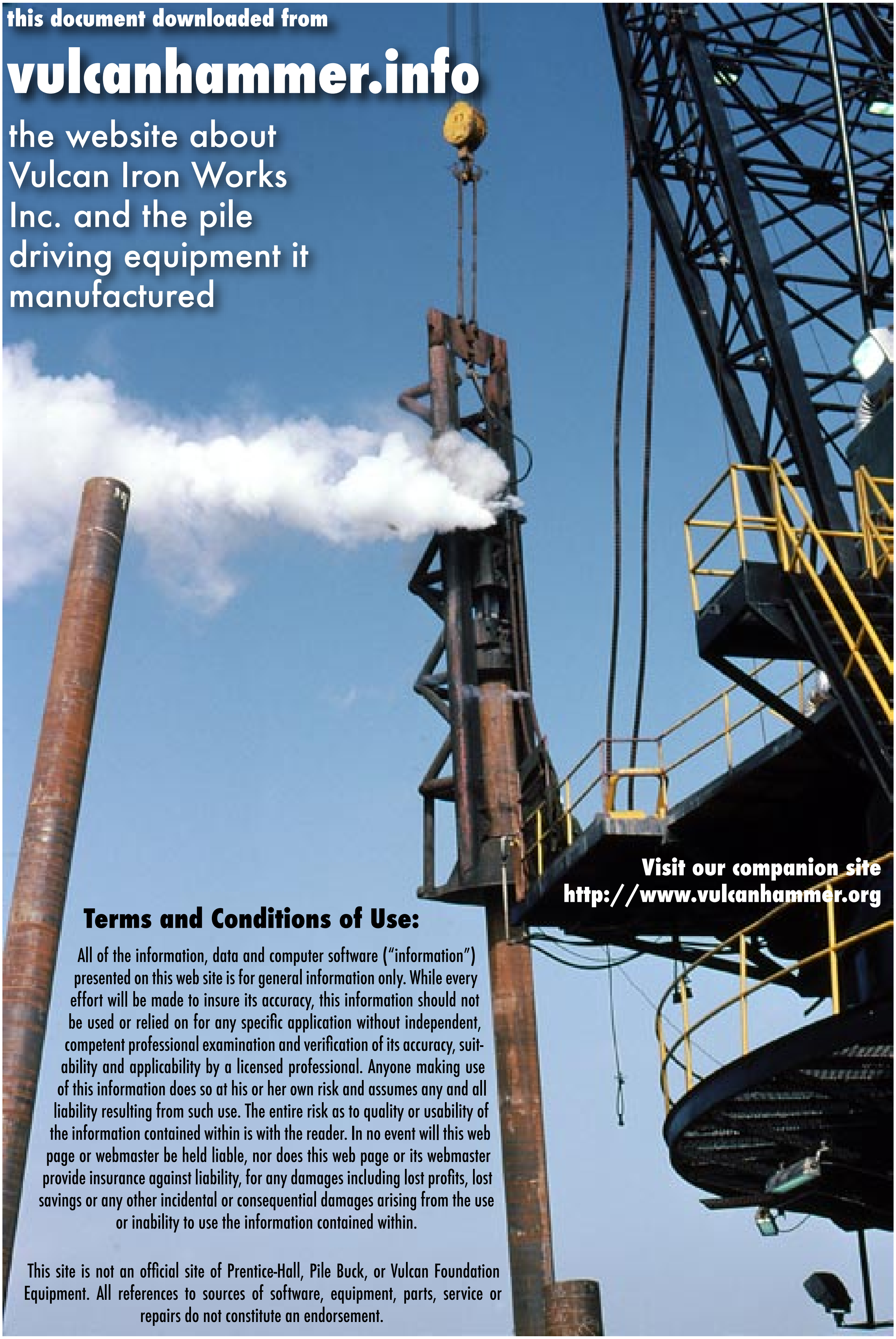

Anchored Sheet Pile Wall Analysis Using Fixed End Method Without Estimation of Point of Contraflexure

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Abstract

The fixed end method used for the design of anchored sheet pile walls has been used with success since before World War II; however, computational limitations have forced designers to use simplifications such Hermann Blum developed. The original method called for the use of an "elastic line" solution, where the penetration of the sheet piling below the excavation line was estimated using statically indeterminate beam theory. This paper develops the governing equations for the "elastic line" method for a simple case and presents the solution in two ways: parametrically using charts, and for specific cases using an online computer algorithm. Comparison with other solution techniques is presented, and suggestions for broader applications are made. The adjustment of the penetration for the residual toe load is also discussed, and the limitations of current practice in this adjustment are detailed.

Introduction

When using classical sheet pile methods, anchored sheet pile walls are generally analysed by one of two methods: free or fixed end method, the "end" referring to the pile toe. The selection of the method used is usually unrelated to actual state of end fixity. Both of these methods have been employed for a long time. The two methods are schematically compared in Figure 1, and are described in detail by Warrington and Lindahl (2007).

Free end methods result in a statically determinate structure, but that structure is only stable under certain load conditions, namely a) the sum of the distributed loads equals to the anchored load and b) the passive earth pressure resistance below the excavation line is sufficiently large to prevent overturning. Generally criterion (b) governs, with the resultant anchor load being a result of the calculations.

1 Originally posted on vulcanhammer.info October 2007, in conjunction with the routine described in the paper.

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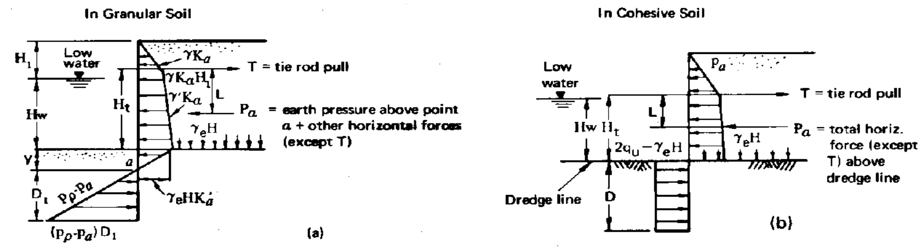
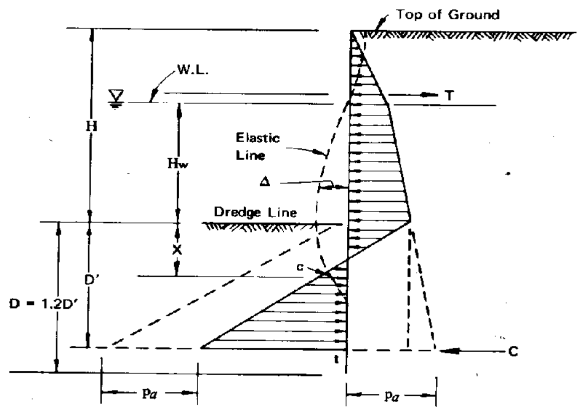


Free Earth Support Method



Fixed Earth Support Method

Figure 1 Schematic for Free and Fixed End Methods (after Lindahl (1974))

With fixed end methods, the system has two supports:

1. A simple support at the anchor.
2. A fixed support at the pile toe (Figure 1, lower figure, point "t".)

Because of the nature of the supports, the system is statically indeterminate, forcing the designer to consider system deflections. A quick glance at the system would lead one to conclude that the embedment length of the sheet piling below the excavation line can be any length; however, an additional criterion is that the moment at the pile toe is zero, irrespective of the fact that the toe is rotationally constrained. In fact, the fixed end method can be conceived with a pinned pile toe whose slope must be zero instead of a fixed end support, but in either case slopes and deflections must be determined.

Free and fixed end methods are mutually independent; their results should not be mixed. Sheet pile walls designed with free end methods tend to be shorter and have a higher moment of inertia against bending stresses than fixed end method designed walls, while the latter tend to be longer with a lower moment of inertia. The moment computed using the free end method can be reduced using Rowe's moment reduction method; this does not apply to free end analysis.

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Outline of the Fixed Earth Method

The fixed end method outlined above is formally referred to as the "elastic line" (see Figure 1, "elastic curve" is more descriptive) method. It is the "ideal" method for solving the problem. The basics of this method are as follows:

1. Determine the earth pressure profile from the soil data, taking into consideration the following:
   a. The distance from the pile head to the excavation line,
   b. the first estimate of the optimum location of the anchor,
   c. the location of the water table (and unbalanced hydrostatic forces if applicable,) and
   d. a first estimate of a suitable sheet pile section, which gives a trial value of the moment of inertia, section modulus and modulus of elasticity.

      Any factor of safety or load and resistance factor should be applied to the earth pressures first.
2. Make a first estimate of the depth. Determine if the moment at the fixed end is zero. If this is not the case, vary the depth until this is achieved.
3. Increase the length of the sheeting below the computed penetration to account for the reaction load at the pile toe, which is necessary since there is generally no "fixity" at the pile toe.
4. Determine the maximum bending moment in the sheeting. Check the resulting bending stress against the failure criteria for the material of the sheeting. If this is not satisfactory (stress is too high or low,) change the section. Transverse bending (Warrington and Lindahl, 2007) is not considered in this paper.
5. Determine the maximum deflection of the sheeting. For ferrous sheeting, this is generally not necessary. For non-ferrous sections, due to their lower values of moment of inertia and modulus of elasticity, this is more important.

The critical step is (2). Since the beam described by the fixed end moment method is simply a cantilever beam with a simple support, at first glance this would look to be a simple matter; however, the complex nature of the soil loading, even for simple soil profiles, and the need to iterate the solution made traditional hand calculations laborious.

This difficulty has been recognised from the beginning of fixed end analysis. One of the advantages of the free end method is that the system, while theoretically unstable, is statically determinate. Is it possible to reduce a fixed end model to a simpler, statically determinate system?

One of the first attempts to solve this problem was undertaken by Hermann Blum (Tschebotarioff, 1951). The idea behind his solution is shown in Figure 2.

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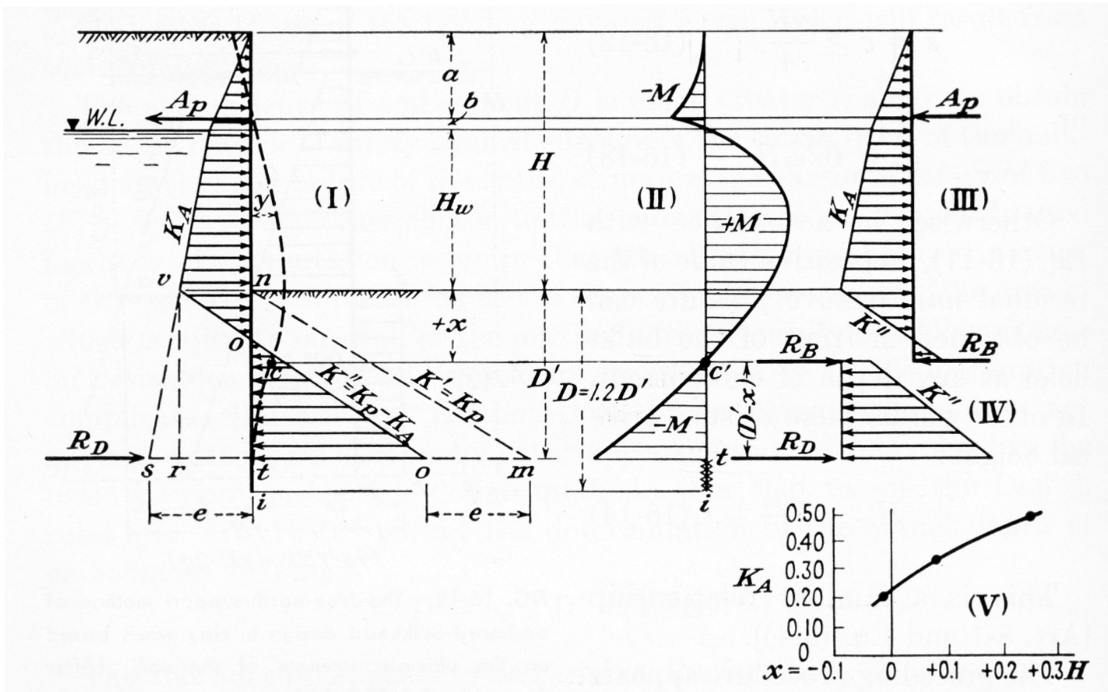

Figure 2 Basic Concept of Blum's Method (after Tschbotarioff, 1951)

Figure 2(I) shows a simple fixed end sheet profile with the elastic line of the sheeting as it is deflected under the load. Just under the excavation line is the point of contraflexure *c'* (Figure 2 (II)), the point where the curvature of the sheet is reversed from its state below the support (convex with respect to the active pressures) to concave with respect to the same side. If one were to take the derivative of the slope equation, the point of contraflexure, being the point at which the rate of change of the slope changes from positive to negative, is the zero point of that derivative curve. Since the moment is the first derivative of the slope multiplied by the product of the modulus of elasticity and the moment of inertia, the moment at this point should be zero, without consideration of integration constants.

Blum reasoned that a simple support could be added at this point. His idea was to divide the beam into two hinged, statically determinate parts, one above the point of contraflexure and one below. Both of these beams would be simply supported, more readily soluble than the statically indeterminate structure. In this way the fixed earth support method became possible to solve using hand calculations, whether these be purely mathematical or graphically solved. His method has come to be referred to as the *equivalent beam method*, although any sheet piling design problem is in fact a solution to a beam problem.

Blum's determination of the point of contraflexure was based on model tests conducted in Germany a few years before he first presented his method in 1931. It is shown in Figure 2(V), and is a function of the active earth pressure coefficient. The distance $+x$ from the excavation line to the point of contraflexure is shown as a ratio of $+x$ to H, the distance from the pile head to the excavation line.

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Tschebotarioff expressed reservations about the derivation of this point of contraflexure due to the nature of the laboratory tests. He developed a method that moved the point of contraflexure to the excavation line, using specially derived coefficients. Although this simplified Blum's Method, the values of the coefficients he proposed were not fully quantified.

An alternative method of determining the point of contraflexure is shown by Arbed (1986). Here the point of contraflexure is assumed to be the point where the net earth pressure changes direction, i.e., the point O in Figure 2(I). This method was also recommended by the British Steel Corporation (1984).

Once the location of the point of contraflexure is determined, the reaction at the support and the "reaction" at the point of contraflexure can be computed. The latter reaction is used for the lower beam. Its length is determined by summing moments for this beam about the pile toe, taking into consideration the net earth pressure of this portion of the beam (Figure 2(IV)). The location at which the moment sum is zero is the target length of this beam.

Unfortunately, with this determined, summing forces for this beam will reveal a reaction R_D at the toe of the pile. This reaction has no real resistance in the soil; therefore, additional length must be added to the sheet pile wall to account for this reaction. A general "rule of thumb" states that this additional length is 20% of the computed embedment of the sheet pile wall. This problem will be examined in more detail later.

Blum's Method and those which follow were developed to adapt the fixed end method to computational possibilities of their day. Is it possible to eliminate the estimate of the point of contraflexure and thus obtain a superior (to say nothing of a more uniform) result?

The object of this paper is to show that this is practical. The method employed is to use a closed form solution for the problem for a simple yet common case: a uniform cohesionless soil with the water table between the excavation line and the support. In the process, a dimensionless solution to the problem is developed similar to that shown in chart form for the free end method (Figure 3.) This in turn can be applied to academic use for solutions to simple sheet piling wall problems.

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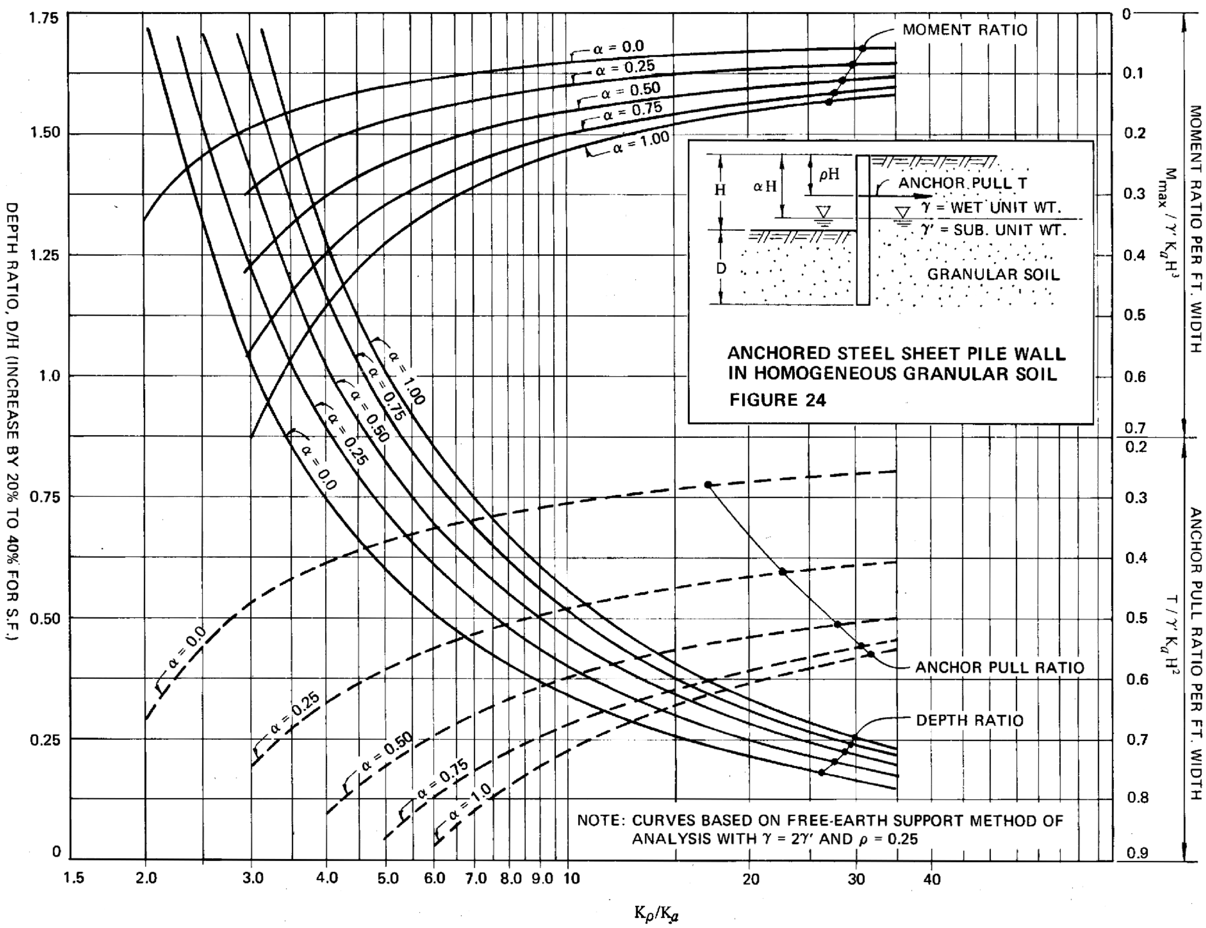

Figure 3 Free End Support Method Parametric Results (after Lindahl, 1974)

Development of the Method

Let us consider the system shown in Figure 4. For simplicity and ease of layout, the sheet pile system will be drawn “ on its side. ”

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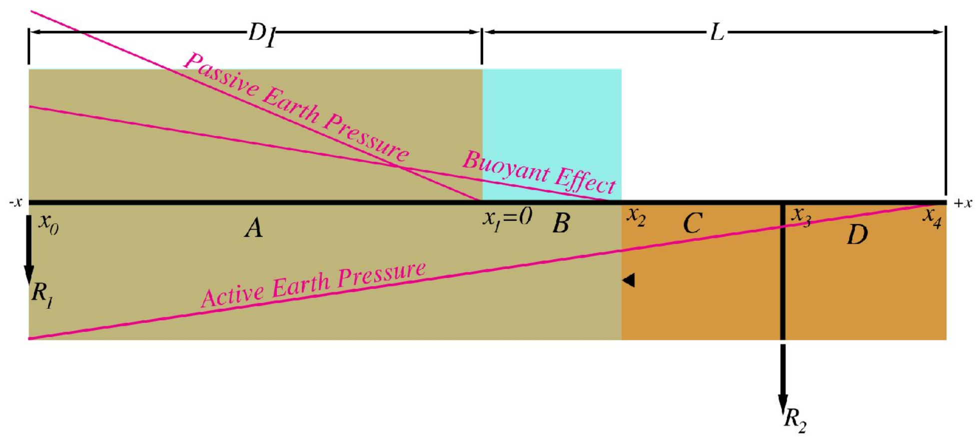

Figure 4 Sheet Pile System for Fixed Earth Support Method

At this point, some judicious selection of coordinates and variables is necessary to simplify the results:

1. The origin of the system is at the excavation line (x_1).
2. The ratios α and β are also taken from the excavation line. This is different from the system shown in Figure 3. These are expressed mathematically as

$$\beta = \frac{x_2}{L}$$

Eq. 1

and

$$\alpha = \frac{x_3}{L}$$

Eq. 2

where L is the distance from the excavation line to the pile head and x_2 and x_3 are the distances from the excavation line to the water table and the support respectively. It is assumed that

$$0 \geqslant \alpha \geqslant \beta \geqslant 1$$

Eq. 3

There are five " critical " points, which in turn define four regions. Within each region the loading profile is continuous. This division is important for the integration process.

The integrations and related algebraic operations were performed using Maple software running on a Macintosh G4 PowerPC computer.

There are three separate lateral earth pressure forces assumed on the pile. The first is the active earth pressure force on the entire length of the pile, expressed as

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$$\sigma_1 = K_a \gamma_1 (L - x)$$
Eq. 4

where

- σ_1 = active earth pressure without consideration of buoyant force of the water on the soil, kPa
- γ_1 = saturated unit weight of the soil, kN/m^3
- K_a = active earth pressure coefficient
- x = coordinate of point along sheet pile wall, m

For the purpose of simplicity, we will assume, as is the case in Figure 3, that

$$\gamma_0 = \frac{\gamma_1}{2}$$
Eq. 5

where γ_0 = submerged unit weight of the soil, kN/m^3.

In this case, Eq. 4 can be expressed as

$$\sigma_1 = 2 K_a \gamma_0 (L - x)$$
Eq. 6

The second force is the buoyant force of the water, which is expressed as a " passive earth pressure " under the phreatic surface compensating the active pressure. This is expressed as

$$\sigma_2 = -K_a \gamma_0 (\beta L - x), x \leqslant x_2$$
Eq. 7

The third is the passive earth pressure, which for this case is always submerged,

$$\sigma_3 = K_p \gamma_0 x, x \leqslant 0$$
Eq. 8

Defining

$$\kappa = \frac{K_p}{K_a}$$
Eq. 9

Eq. 8 can be stated as

$$\sigma_3 = \kappa K_a \gamma_0 x, x \leqslant 0$$
Eq. 10

The use of κ makes it possible to apply design factors (ASD or LFRD) to the model easily.

At this point we should also define

$$\rho=\frac{D_1}{L}=\frac{-x_0}{x_4}$$

Eq. 11

where D_1 is the distance from the excavation line to the pile toe. Determination of ρ is obviously the central objective of the fixed earth support method.

Now that we have defined our basic earth pressure forces, the first task is to develop expressions for the reactions at the supports (the anchor and pile toe.) By integrating Eq. 6, Eq. 7 and Eq. 10 to determine the resultant forces, and establishing the location of those resultant forces, we can develop two equations: the equation for the equilibrium of the forces for the system and the moments about the pile toe. Solving these for the two reactions yields the following expressions:

$$R_1=K_a\gamma_0 L^2\frac{6\beta\rho\alpha+3\beta\rho^2-6\rho^2-2\rho^3+2-\beta^3-6\alpha-12\rho\alpha-3\rho^2\alpha+3\beta^2\alpha+3\beta^2\kappa\alpha}{6(\alpha+\rho)}$$

Eq. 12

$$R_2=K_a\gamma_0 L^2\frac{3\beta\rho^2+\rho^3\kappa+\beta^3+3\beta^2\rho-6\rho^2-\rho^3-2-6\rho}{6(\alpha+\rho)}$$

Eq. 13

From this point the next step is to determine the moment and deflection profile of the sheet pile wall. Although many different schemes have been employed to do this, straight beam integration was chosen, as it resulted in a complete profile along the length of the beam. The first step was to integrate from Eq. 6, Eq. 7 and Eq. 10 again, but this time with a different purpose. Starting with Region A, the pile toe reaction (R_1) is a boundary condition at x_0 for the shear. Integrating the load equations results in a shear profile for Region A, which ends with a shear value at x_1. This in turn becomes the boundary condition for Region B, and the integration continues, only this time using only Eq. 6 and Eq. 7 as the passive pressures do not act above the excavation line. This process is repeated for the other two regions with one important addition: the support reaction (R_2) represents the change in shear between Regions C and D rather than a simple boundary condition.

Development of the shear profile is interesting, but the most critical profiles are the moment and deflection profile. The moment profile can be developed by repeating the process used for the shear profile, using the zero moment condition at x_0 as a boundary condition and the same condition at x_4 as a check. The equations for the moment for the four regions are as follows:

$$M_A=K_a\gamma_0 L^3\frac{\rho L+x}{6(\alpha+\rho)}(3L^2\beta^2\alpha+L^2\rho^2\kappa\alpha+2L^2-6L^2\rho\alpha+3L^2\beta\rho\alpha-6L^2\alpha-L^2\beta^3$$
$$-L^2\rho^2\alpha-L\rho^2\kappa x+L\rho^2 x-L\rho x\kappa\alpha+6Lx\rho-3\beta Lx\rho+Lx\rho\alpha+6Lx\alpha-3\beta Lx\alpha$$
$$+\kappa x^2\rho-x^2\rho+\kappa x^2\alpha-x^2\alpha)$$

Eq. 14

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$$M_B = K_a \gamma_0 \frac{1}{6(\alpha+\rho)}(3L^3\beta^2\alpha\rho + \kappa\rho^3L^3\alpha + 2L^3\rho - 6\rho^2L^3\alpha + 3\beta L^3\rho^2\alpha - 6L^3\alpha\rho - L^3\beta^3\rho$$
$$-\rho^3L^3\alpha - L^2\rho^3\kappa x - 3L^2\beta\rho^2x + 3L^2\beta^2\alpha x - 6L^2\alpha x - L^2\beta^3x + L^2\rho^3x + 2L^2x + 6\rho^2L^2x$$
$$+6Lx^2\alpha + 6Lx^2\rho - x^3\alpha - x^3\rho - 3\beta Lx^2\alpha - 3\beta Lx^2\rho)$$

Eq. 15

$$M_c = K_a \gamma_0 \frac{1}{6(\alpha+\rho)}(6Lx^2\alpha - L^2\beta^3x + \beta^3L^3\alpha - 2x^3\rho + 3L^3\beta^2\alpha\rho - 6L^3\alpha\rho - 6\rho^2L^3\alpha + \kappa\rho^3L^3\alpha$$
$$+3\beta L^3\rho^2\alpha + 2L^3\rho - \rho^3L^3\alpha + L^2\rho^3x - 6L^2\alpha x - 2x^3\alpha + 2L^2x - L^2\rho^3\kappa x - 3L^2\beta\rho^2x - 3L^2\beta^2\rho x$$
$$+6Lx^2\rho + 6\rho^2L^2x)$$

Eq. 16

$$M_D = K_a \gamma_0 \frac{1}{3}(L-x)^3$$

Eq. 17

It should be noted that, if we define

$$x = x'L$$

Eq. 18

and make the appropriate substitutions, Eq. 14, Eq. 15, Eq. 16 and Eq. 17 can all be written in the form

$$M_n = K_a \gamma_0 L^3 A(\beta, \alpha, \rho, x')$$

Eq. 19

where $A(\beta,\alpha,\rho,x')$ is a dimensionless coefficient. This enables us to write the moments in dimensionless form as a ratio of a " nominal " moment $K_a \gamma_0 L^3$, and thus generalise the results.

To obtain the deflections, another two rounds of integration are necessary. In integrating from moment to slope, it is necessary to additionally divide by the product of the modulus of elasticity and the moment of inertia. Doing this and substituting the appropriate boundary conditions, the deflections are

$$d_A = K_a \gamma_0 \frac{(\rho L+x)^3}{360(\alpha+\rho)EI}(-30\rho^2L^2 - 8L^2\rho^3 + 20L^2 + 18L^2\rho^2\kappa\alpha + 15L^2\beta\rho^2 - 90L^2\rho\alpha - 60$$
$$-10L^2\beta^3 + 8L^2\rho^3\kappa - 18L^2\rho^2\alpha + 45L^2\beta\rho\alpha + 30L^2\beta^2\alpha - 9L\rho^2\kappa x + 9L\rho^2x - 9L\rho x\kappa\alpha$$
$$-15\beta Lx\rho + 9Lx\rho\alpha + 30Lx\rho - 15\beta Lx\alpha + 30Lx\alpha + 3\kappa x^2\rho - 3x^2\rho - 3x^2\alpha + 3\kappa x^2\alpha)$$

Eq. 20

$$d_B = K_a \gamma_0 \frac{1}{360(\alpha+\rho) E I} (10 L^2 \rho^3 x^3 + 30 L x^4 \rho + 60 \rho^2 L^2 x^3 + 60 \rho^2 L^4 x - 60 \rho^4 L^4 x - 15 \rho^5 L^4 x + 30 L x^4 \alpha + 60 L^3 \rho x^2 - 60 L^2 \alpha x^3 - 3 x^5 \rho - 10 L^2 \beta^3 x^3 + 120 \rho^3 L^4 \beta \alpha x + 90 \rho^2 L^4 \beta^2 \alpha x + 45 \rho^4 L^4 \kappa \alpha x - 240 \rho^3 L^4 \alpha x + 30 \rho^4 L^4 \beta x + 90 L^3 \beta^2 \alpha \rho x^2 - 180 \rho^2 L^4 \alpha x - 15 \beta L x^4 \rho + 90 \beta L^3 \rho^2 \alpha x^2 - 15 \beta L x^4 \alpha + 30 L^2 \beta^2 \alpha x^3 - 30 \rho^3 L^3 \alpha x^2 - 30 L^2 \beta \rho^2 x^3 - 180 \rho^2 L^3 \alpha x^2 - 10 L^2 \rho^3 \kappa x^3 - 30 L^3 \beta^3 \rho x^2 + 30 \kappa \rho^3 L^3 \alpha x^2 - 180 L^3 \alpha \rho x^2 + 20 L^2 x^3 - 3 x^5 \alpha + 15 \rho^5 L^4 \kappa x - 30 \rho^2 L^4 \beta^3 x - 45 \rho^4 L^4 \alpha x - 30 \rho^5 L^5 - 8 \rho^6 L^5 + 20 \rho^3 L^5 + 18 \rho^5 L^5 \kappa \alpha + 15 \rho^5 L^5 \beta - 90 \rho^4 L^5 \alpha - 60 \rho^3 L^5 \alpha - 10 \rho^3 L^5 \beta^3 + 8 \rho^6 L^5 \kappa - 18 \rho^5 L^5 \alpha + 45 \rho^4 L^5 \beta \alpha + 30 \rho^3 L^5 \beta^2 \alpha)$$

Eq. 21

$$d_C = K_a \gamma_0 \frac{1}{360(\alpha+\rho) E I} (3 \beta^5 L^5 \alpha + 3 L^5 \beta^5 \rho + 10 L^2 \rho^3 x^3 + 30 L x^4 \rho + 60 \rho^2 L^2 x^3 + 60 \rho^2 L^4 x - 60 \rho^4 L^4 x - 15 \rho^5 L^4 x + 30 L x^4 \alpha + 60 L^3 \rho x^2 - 60 L^2 \alpha x^3 - 6 x^5 \rho - 10 L^2 \beta^3 x^3 + 120 \rho^3 L^4 \beta \alpha x + 90 \rho^2 L^4 \beta^2 \alpha x + 45 \rho^4 L^4 \kappa \alpha x - 240 \rho^3 L^4 \alpha x + 30 \rho^4 L^4 \beta x + 90 L^3 \beta^2 \alpha \rho x^2 - 180 \rho^2 L^4 \alpha x + 90 \beta L^3 \rho^2 \alpha x^2 - 30 \rho^3 L^3 \alpha x^2 - 30 L^2 \beta \rho^2 x^3 - 180 \rho^2 L^3 \alpha x^2 - 10 L^2 \rho^3 \kappa x^3 + 30 \kappa \rho^3 L^3 \alpha x^2 - 180 L^3 \alpha \rho x^2 + 20 L^2 x^3 - 6 x^5 \alpha + 15 \rho^5 L^4 \kappa x - 30 \rho^2 L^4 \beta^3 x - 45 \rho^4 L^4 \alpha x + 30 \beta^3 L^3 \alpha x^2 - 30 L^2 \beta^2 \rho x^3 - 15 \beta^4 L^4 \rho x - 15 \beta^4 L^4 \alpha x - 30 \rho^5 L^5 - 8 \rho^6 L^5 + 20 \rho^3 L^5 + 18 \rho^5 L^5 \kappa \alpha + 15 \rho^5 L^5 \beta - 90 \rho^4 L^5 \alpha - 60 \rho^3 L^5 \alpha - 10 \rho^3 L^5 \beta^3 + 8 \rho^6 L^5 \kappa - 18 \rho^5 L^5 \alpha + 45 \rho^4 L^5 \beta \alpha + 30 \rho^3 L^5 \beta^2 \alpha)$$

Eq. 22

$$d_D = K_a \gamma_0 \frac{1}{360 E I} (30 \rho^2 L^5 \alpha \beta^2 + 30 \rho^3 L^5 \alpha \beta - 10 \rho^2 L^5 \beta^3 + 15 \rho^4 L^5 \beta - 60 L^4 \alpha x - 15 \rho^4 L^4 x - 60 L^4 \rho^3 x + 60 L^4 \rho x - 30 \rho^3 L^4 \alpha x - 180 \rho^2 L^4 \alpha x - 60 L^2 x^3 + 10 L^5 \rho^4 \kappa \alpha + 10 \beta^3 L^5 \rho \alpha - 10 L^5 \kappa \rho^3 \alpha^2 + 20 L^5 \alpha^2 - 30 L^5 \beta^2 \alpha^2 \rho + 20 L^5 \rho^2 - 30 \rho^2 L^5 \alpha^2 \beta + 60 L^5 \rho \alpha^2 - 20 L^5 \rho \alpha - 10 \beta^3 L^5 \alpha^2 + 60 \rho^2 L^5 \alpha^2 + 10 \rho^3 L^5 \alpha^2 - 30 \beta^3 L^4 \rho x + 90 \rho^2 L^4 \alpha \beta x - 6 x^5 + 3 L^5 \beta^5 + 30 L^4 \kappa \rho^3 \alpha x + 30 \beta^3 L^4 \alpha x - 180 L^4 \rho \alpha x + 90 L^4 \beta^2 \alpha \rho x - 30 L^5 \rho^4 + 30 L^4 \rho^3 \beta x + 15 L^4 \rho^4 \kappa x - 15 L^4 \beta^4 x + 8 L^5 \rho^5 \kappa - 60 L^5 \rho^2 \alpha + 30 L x^4 - 8 \rho^5 L^5 - 10 \rho^4 L^5 \alpha - 60 \rho^3 L^5 \alpha + 60 L^3 x^2)$$

Eq. 23

where d_n are the deflections in the given regions in metres and E and I are the modulus of elasticity and moment of inertia per length unit of wall in Pa and m^4, respectively.

As is the case with the moments, if we substitute Eq. 18 into Eq. 20 through Eq. 23, the latter can be written in the form

$$d_n = K_a \gamma_0 L^5 B(\beta, \alpha, \rho, x')$$

Eq. 24

where again $B(\beta,\alpha,\rho,x')$ is a dimensionless coefficient.

The only variable left unknown is ρ. For the integration of both the slope and the deflection equations, the boundary conditions for both of these at x_0 is zero. However, there is no guarantee that the deflection will be zero at the other support, x_3, unless ρ—the last unknown—has a value such that this deflection is zero. Substituting $x = x_3$ into either Eq. 22 or Eq. 23 and equating to zero, the quantity ρ is the solution of the equation

$$\begin{gathered}8(\kappa-1)\rho^5+5(3\beta-6-5\alpha+5\kappa\alpha)\rho^4\\ -20(6\alpha-3\beta\alpha+\alpha^2-\kappa\alpha^2)\rho^3\\ +10(3\beta^2\alpha-12\alpha^2+6\beta\alpha^2-6\alpha+2-\beta^3)\rho^2\\ +20(2\alpha-6\alpha^2-\alpha\beta^3+3\beta^2\alpha^2)\rho\\ -60\alpha^3+20\alpha^2+20\beta^3\alpha^2-6\alpha^5-15\beta^4\alpha+3\beta^5+30\alpha^4=0\end{gathered}$$

Eq. 25

The solution of this equation will be outlined as a part of the discussions of the solution of the problem in general.

Parametric Solution

Now that the basic equations are defined, they can be applied to an actual solution. There are two types of solutions that can be done, either a specific solution for a single case or a parametric solution. The parametric solution will be considered first.

The existence of a parametric solution is suggested by Eq. 19 and Eq. 24. For the free end method, Figure 3 presupposes the existence of a parametric solution. Based on the equations and figure, let us define the following dimensionless variables:

$$\bar{T}=\frac{R_2}{K_a\gamma_0 L^2} \quad \text{(Dimensionless Support Load)}$$

Eq. 26

$$\bar{M}=\frac{M}{K_a\gamma_0 L^3} \quad \text{(Dimensionless Moment)}$$

Eq. 27

$$\bar{d}=\frac{d}{K_a\gamma_0 L^5} \quad \text{(Dimensionless Deflection)}$$

Eq. 28

The Maple software was used to perform the algebra and integration to yield the equations of shear, reactions, moment and deflection. It is possible to use the same software to perform the numerical computations; however, it was decided to construct a php routine run on a web server to perform these calculations, which made the iterative structure of the program simpler, along with greater control of the output format.

This being the case, the greatest challenge became the transferral of the formulas from Maple to php. This process was significantly simplified by the fact that Maple is capable of generating C code for the algebraic expressions it develops. For mathematical expressions, the syntax for C is virtually identical to php. It was thus a matter of calling for the development of the C code, assigning a function for each expression, copying and pasting the code for the function, an adding a dollar sign at the start of each variable (php requires this as C does not.)

Once these functions were set up, they were applied to a range of values of α = 0, 0.25, 0.5, 0.75 and 1. Values for β were the same, but when matched with values of α they were restricted by Eq. 3. Values of κ ranged from 2 to 35 as suggested by Figure 3. To obtain dimensionless results, the values of K_a, γ_o and L were set to unity.

For each case considered, once the data was input the first task of the program was to compute the unadjusted value of ρ. "Unadjusted" means that the penetration of the sheeting is not extended to compensate for the reaction at the toe, which will be discussed in the next section. This is to allow comparison with the values from the free end method as shown in Figure 3. The unadjusted value of ρ is computed using Eq. 25, which is only a function of α, β and κ. This equation is solved using the method of bisection (Chapra and Canale, 1984), bracketed by values of ρ = 0 and ρ = 5. Applying this method to cases of α = 0 and α = 0.25 resulted in a lack of proper convergence, so these cases were not considered further. (As a practical matter, it is unlikely that the support would be placed more than halfway down to the excavation line from the pile head.)

Turning to the remaining cases, these were analysed to establish proper value of ρ, which was consistently established within twenty iterations. Once this was done, the next step was to determine the maximum dimensionless moments and deflections for each case, along with the dimensionless support reaction. The obvious method to do this is to take the derivative of the moment and deflection equations and determine the zero values. This method was employed in Maple to check the results of the php code for specific cases; however, the weakness to this method is that it is necessary to do this for each zone and then determine the maximum from the resulting zone values.

A more sensible solution was to select many evenly-spaced points (200 were chosen for this study,) compute the moments and deflections at each point, and then select the maximum absolute value of moment and deflection from the database. Although the absolute maximum may be missed, errors in solving for the maxima and minima are not without difficulties of their own given the complexity of the equations.

Results of the Parametric Study

The results are presented in graphical form in Figure 5 through Figure 10. All of the plots are log-log scaled.

Anchored Sheet Pile Wall Analysis Using Fixed End Method Without Estimation of Point of Contraflexure

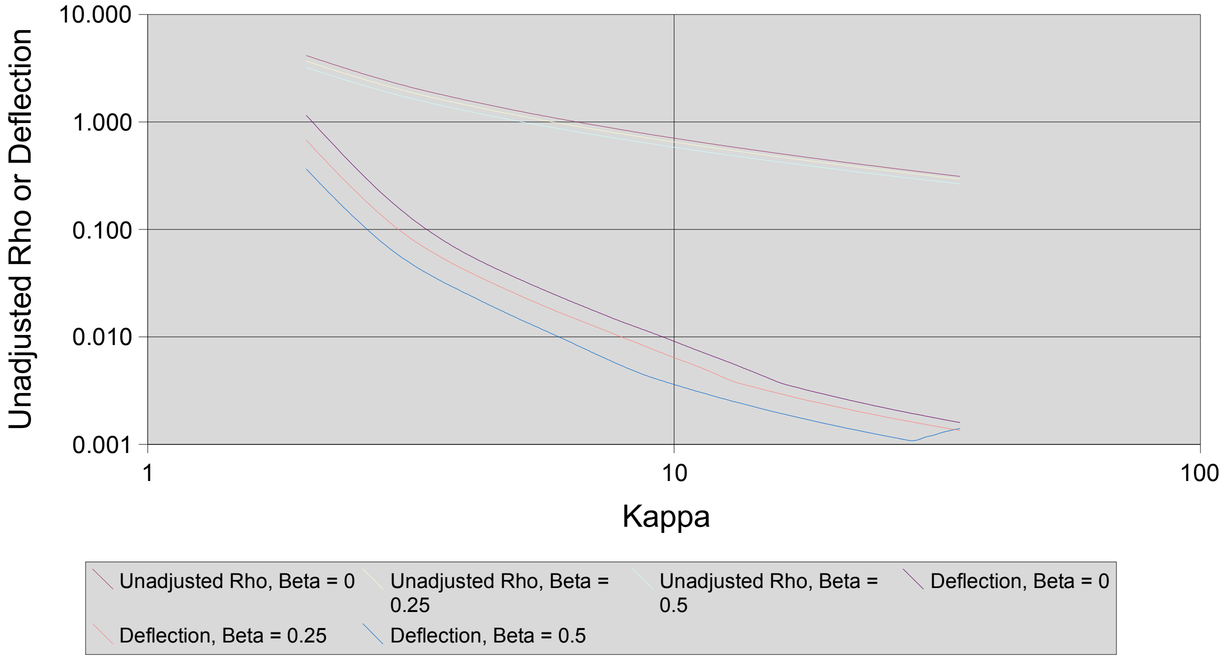

Figure 5 Unadjusted ρ or Deflection, α = 0.5

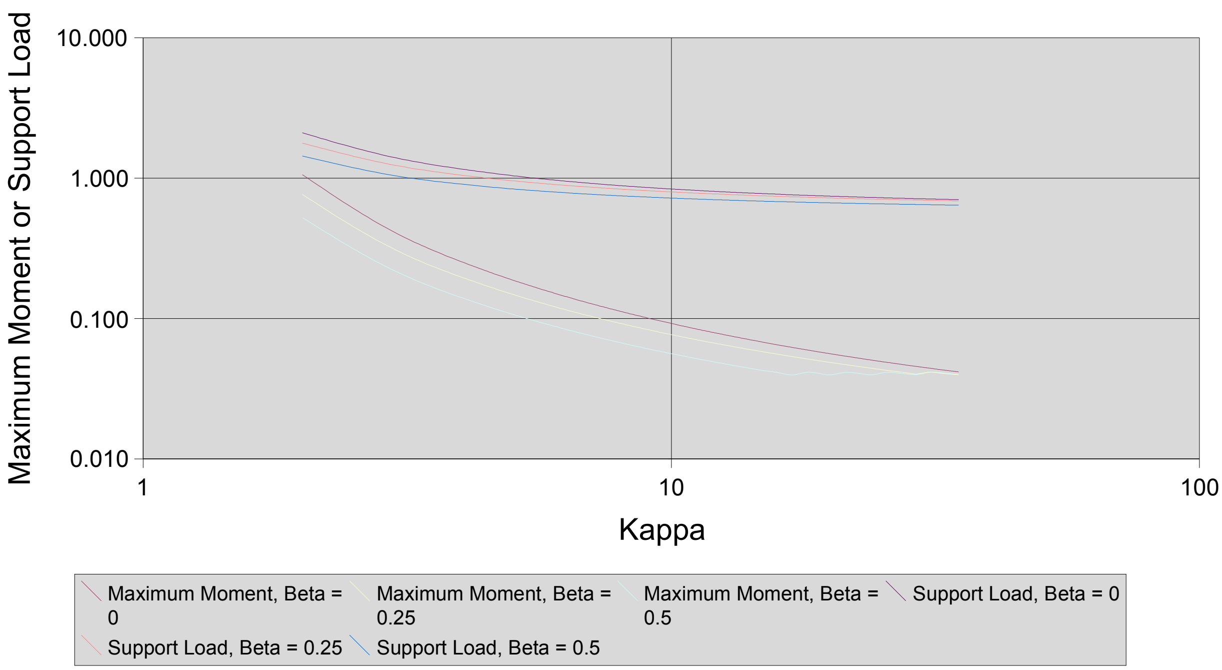

Figure 6 Maximum Moment or Support Load, α = 0.5

Anchored Sheet Pile Wall Analysis Using Fixed End Method Without Estimation of Point of Contraflexure

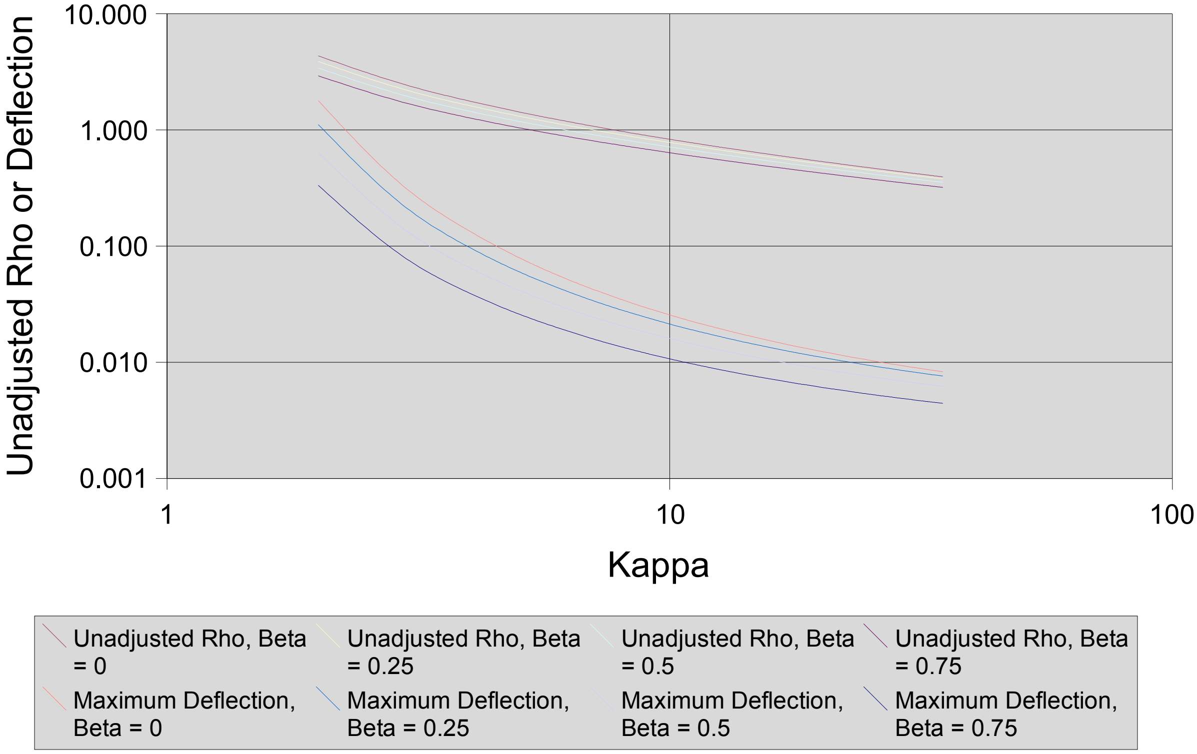

Figure 7 Unadjusted ρ or Deflection, α = 0.75

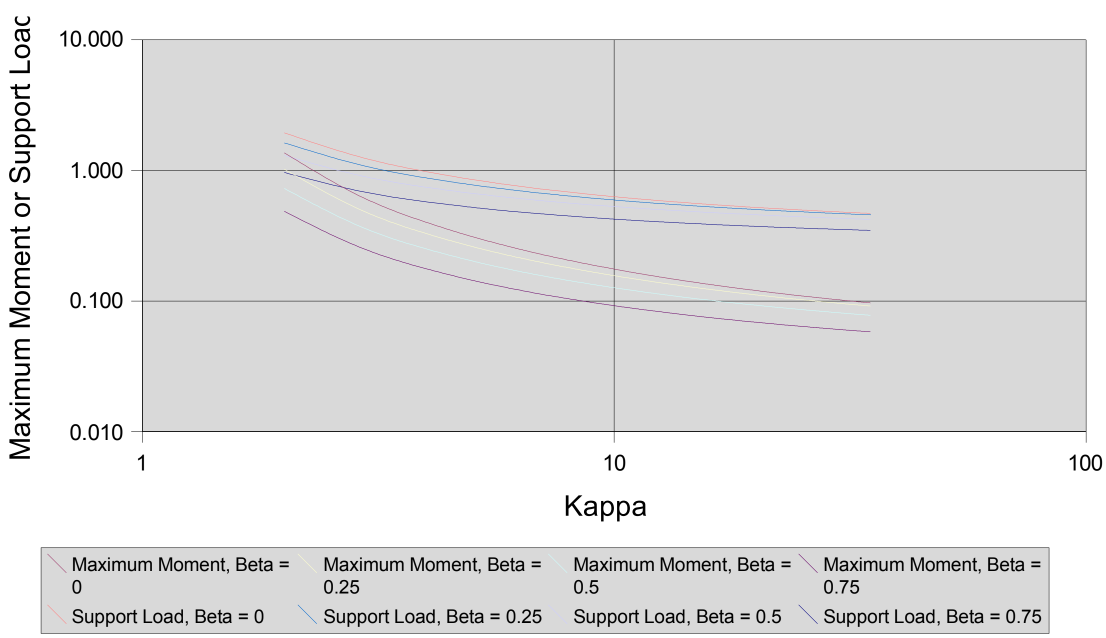

Figure 8 Maximum Moment or Support Load, α = 0.75

Anchored Sheet Pile Wall Analysis Using Fixed End Method Without Estimation of Point of Contraflexure

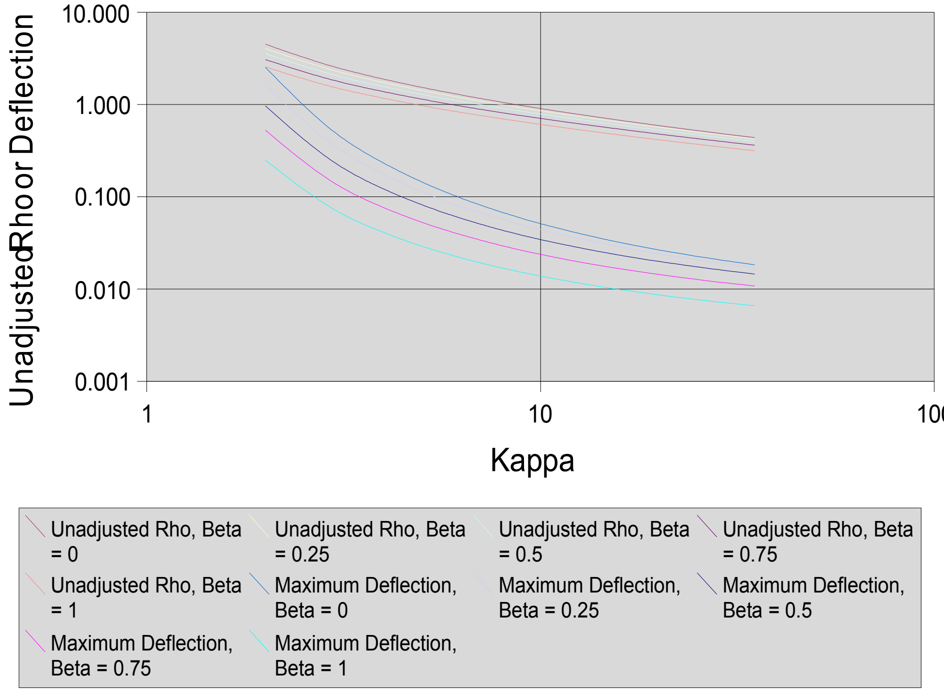

Figure 9 Unadjusted ρ or Deflection, α = 1

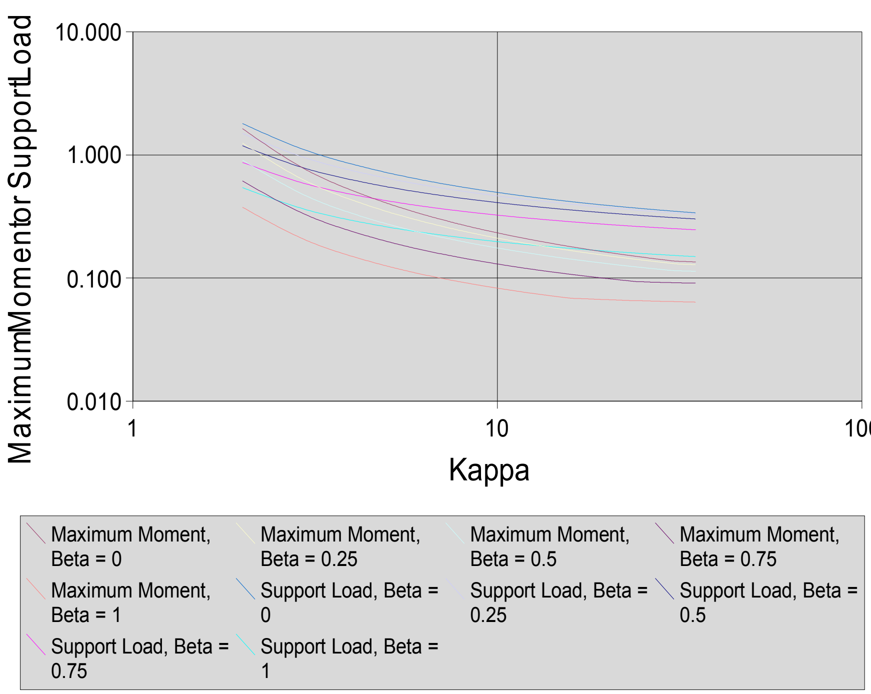

Figure 10 Maximum Moment or Support Load, α = 1

Some *general* observations about the results are as follows:

1. All of the quantities (load, moment, deflection and penetration of the sheeting) tended to decrease with either κ or β. This is because both of these reflect either a relative increase in the active pressure or decrease in the passive pressure.
2. All of the quantities also tend to increase with α, but not at the same rate. The increase in moment and deflection are very marked, probably because of the longer unsupported length of the beam below the support. The increase in support load is not as marked, and the increase in ρ is the least of all.

The use of the parametric method is good to obtain an overview of the general results of the method. As was shown in Figure 3, it is possible to use the charts (or tables with the same data) to determine the results of the method. However, a more practical solution exists, and that is to use the same base computer code to obtain results for specific cases. But before the parametric solution is left, there is one more important issue to consider.

Computation of the Adjustment Factor for ρ

Once the value of ρ (and thus the theoretical penetration of the pile below the excavation line) is computed, it is necessary to adjust this value upward. There is a good deal of confusion concerning the nature of this adjustment. Many engineers look at this adjustment and consider it a "factor of safety" but this is not the case. For the fixed end method, it is a necessary by-product of the method itself.

The fixed end method requires that the toe of the pile have neither slope nor moment, but permits a reaction at the toe. Since soil at the toe is unable to support the reaction, some type of provision for this reaction must be made. If the penetration is continued past the unadjusted point, the net pressure on the extension is driven by the higher passive pressure away from the excavation side of the sheet pile wall. When the net value of the soil force, which increases with depth, reaches the value of the toe reaction, same reaction is balanced. The adjusted value for ρ when this takes place is

$$\rho_{adj} = \frac{3(\beta-2)(\alpha+\rho) \pm \sqrt{3(\alpha+\rho)q}}{3(\kappa-1)(\alpha+\rho)\rho}$$

$$q = 18\alpha + 12\rho - 6\kappa\alpha + 2\kappa + 6\alpha\kappa^2\rho^2 + 12\beta\alpha\kappa\rho - 12\beta\rho + 9\kappa\beta\rho^2 - 2 - 12\beta\alpha + 3\alpha\kappa\beta^2 -$$
$$12\beta\rho\alpha + 18\rho^2 - 9\beta\rho^2 - 10\rho^3\kappa + 6\rho^2\alpha - 12\rho^2\kappa\alpha + 24\rho\alpha + 5\rho^3 + \beta^3 + 3\beta^2\rho - \kappa\beta^3 - 18\rho^2\kappa + 5\kappa^2\rho^3 - 24\kappa\rho\alpha$$

Eq. 29

where ρ_{adj} is the ratio of the length of the sheet pile above the excavation line to the length of the sheet pile below the excavation line after adjustment for the toe reaction. Tschebotarioff (1951) states that, for Blum's Method,

$$\frac{\rho_{adj}}{\rho} = 1.2$$

Eq. 30

This amounts to a 20% increase in the penetration of the wall from its "theoretical" (better unadjusted) value. This factor is very common in fixed earth analyses. But what values would result if the additional penetration were actually computed based on the additional wall length needed to balance the toe reaction?

The answer to this question is suggested in Figure 11.

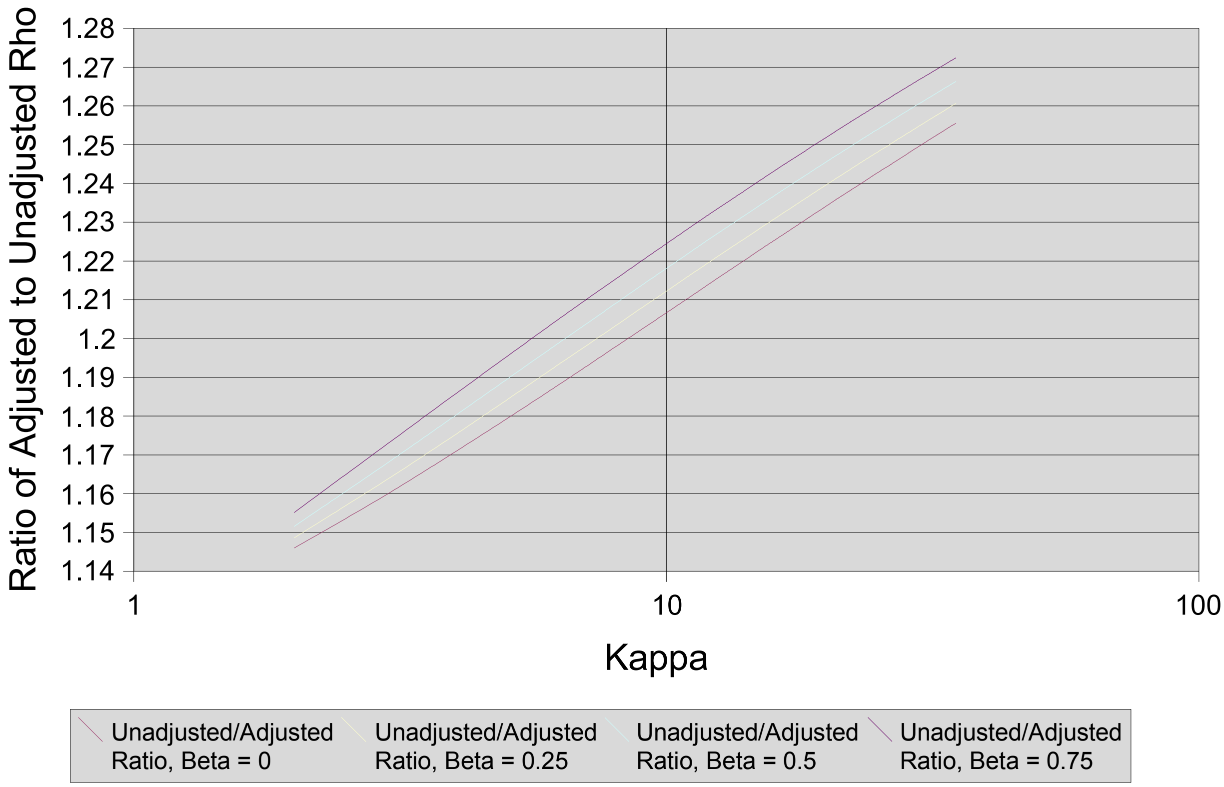

Figure 11 Ratio of ρ_{adj} to ρ, α = 0.75

The figure shows that, as κ increases, the need to adjust the penetration of the sheet pile below the excavation line likewise increases. Figure 3 suggests that this ratio can vary between 1.2 and 1.4 for the free end method, and a similar variation exists for the fixed end method as well.

These results indicate that this factor should be computed based on the actual conditions of the case in question, although the method requires that the unadjusted ρ be computed first. This will be included in the solution algorithm for the specific cases.

Case Specific Solution

The case specific solution uses the same basic algorithm as the parametric solution, but instead of iterating for a range of cases, it takes input data for one specific case and returns the results. This solution is also solved using a php program on a web server, which can than be made widely available for academic use.

The input form, which can accept data in either SI or US units, is shown in Figure 12.

Input Variables for Fixed Earth Method

Units	SI	US	Parameter
Submerged Unit Weight of the Soil	kN/m^3	pcf	
Active Earth Pressure Coefficient	Dimensionless		
Passive Earth Pressure Coefficient	Dimensionless		
Height of Wall Above Excavation Line	m	ft.	
Height of Support Below Wall Top	m	ft.	
Height of Water Table Below Wall Top	m	ft.	
Modulus of Elasticity of Wall Material	GPa	ksi	
Moment of Inertia of Wall Section per unit length of wall	cm^4/m	in^4/ft	
Submit Sheet Pile Data	Reset		

Figure 12 Input Variables for Fixed Earth Method

To test the specific solution, two cases were developed, one in US units and the other in SI units. The parameters of the solution, as they actually appear in the input variables table, are shown in Figure 13 and Figure 14 respectively.

Input Variables for Fixed End Sheet Pile Wall Analysis

Parameter	Units	Value
Submerged unit weight of soil	pcf	60
Height of wall above excavation line	ft.	10
Distance of support below top of wall	ft.	2.5
Distance of water table below top of wall	ft.	5
Alpha	Dimensionless	0.75
Beta	Dimensionless	0.5
Active earth pressure coefficient	Dimensionless	0.333
Passive earth pressure coefficient	Dimensionless	3
Kappa	Dimensionless	9
Modulus of elasticity for sheet pile material	ksi	30000
Moment of inertia of sheet pile section	in^4/ft.	84.4

Figure 13 Input Variables for Fixed End Analysis, US Units

Anchored Sheet Pile Wall Analysis Using Fixed End Method Without Estimation of Point of Contraflexure

Input Variables for Fixed End Sheet Pile Wall Analysis

Parameter	Units	Value
Submerged unit weight of soil	kN/m^3	10.8
Height of wall above excavation line	m	7
Distance of support below top of wall	m	1
Distance of water table below top of wall	m	5
Alpha	Dimensionless	0.857
Beta	Dimensionless	0.286
Active earth pressure coefficient	Dimensionless	0.271
Passive earth pressure coefficient	Dimensionless	3.688
Kappa	Dimensionless	13.6
Modulus of elasticity for sheet pile material	GPa	200
Moment of inertia of sheet pile section	cm^4/m	13513

Figure 14 Input Variables for Fixed End Analysis, SI Units

The program then returns the following data for every point analysed along the sheet pile wall:

1. Location of point along sheet pile wall.
2. Shear per unit length of wall
3. Moment per unit length of wall
4. Slope
5. Deflection

Since the data is in tabular form, it can be easily inserted into a spreadsheet and subsequently graphed. The tabular data also show the locations of the various maxima. Plotting the tabular data yields the results shown in Figure 15 and Figure 16. The slope and deflection results are plotted on the right y-axis.

Anchored Sheet Pile Wall Analysis Using Fixed End Method Without Estimation of Point of Contraflexure

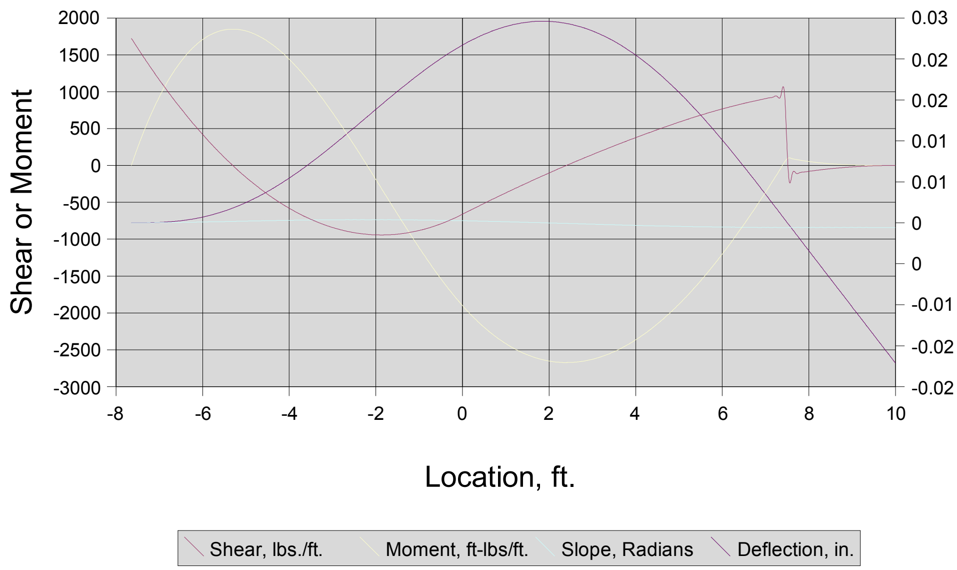

Figure 15 Plotted Results for US Units Case

Figure 16 Plotted Results for SI Units Case

Anchored Sheet Pile Wall Analysis Using Fixed End Method Without Estimation of Point of Contraflexure

The final summary data for the two test cases is shown in Figure 17 and Figure 18 .

Other Results of Sheet Pile Analysis

Parameter	Units	Value
Unadjusted value of rho	Dimensionless	0.765
Number of iterations to compute rho	Dimensionless	16
Unadjusted length of pile penetration below excavation line	ft.	7.65
Pile toe reaction	lbs./ft.	1726.8
Ratio of adjusted to unadjusted pile penetration below excavation line	Dimensionless	1.214
Adjusted length of pile penetration below excavation line	ft.	9.29
Support reaction	lbs./ft.	-1087.3
Extreme Shear	lbs./ft.	1726.8
Extreme Moment	ft-lbs/ft.	2677.1
Extreme Angle	Radians	0.00058
Extreme deflection	in.	0.025

Figure 17 Summary of Results, US Units

Other Results of Sheet Pile Analysis

Parameter	Units	Value
Unadjusted value of rho	Dimensionless	0.666
Number of iterations to compute rho	Dimensionless	16
Unadjusted length of pile penetration below excavation line	m	4.66
Pile toe reaction	kN/m	167.9
Ratio of adjusted to unadjusted pile penetration below excavation line	Dimensionless	1.23
Adjusted length of pile penetration below excavation line	m	5.73
Support reaction	kN/m	-68.6
Extreme Shear	kN/m	167.9
Extreme Moment	kN-m/m	152.9
Extreme Angle	Radians	0.01522
Extreme deflection	mm	39.539

Figure 18 Summary of Results, SI Units

Both of these were also analysed with the Maple V worksheets where the equations were originally derived, and the results generally agree within three significant figures.

Comparison with Other Solutions

The results of the case-specific solution were compared with two other solution techniques.

The first was the CFRAME program, which is a finite-element program developed by the U.S. Army Corps of Engineers. The program was run for both cases, although it was not used to estimate the length of the pile below the excavation line (that was an input variable from Figure 18 and Figure 17.) The simplest way to present the results of this analysis is to compare them using Table 1 and Table 2.

Table 1 Comparison of Results Between Closed Form Solution and CFRAME, US Units Case

Location	***Moment using closed form solution, ft-lbs***	***Moment using CFRAME, ft-lbs***	***Deflection using closed form solution, in.***	***Deflection using CFRAME, in.***
0	1	0	0.000	0.000
1	-1901	-1907	0.022	0.022
2	-1882	-1886	0.016	0.016
3	83	104	0.000	0.000
4	0	0	-0.017	-0.017

Table 2 Comparison of Results Between Closed Form Solution and CFRAME, SI Units Case

Location	***Moment using closed form solution, kN-m***	***Moment using CFRAME, kN-m***	***Deflection using closed form solution, mm***	***Deflection using CFRAME, mm***
0	0.0	0.6	0.0	0.0
1	80.8	80.8	30.2	30.6
2	152.4	152.4	39.4	40.0
3	0.5	1.0	0.1	0.0
4	0.0	0.0	15.3	15.2

The results in both cases show general agreement, although some round-off error (due to the transference of the data) is evident. For locations 1, 2, and 3, the results reported from the closed form solution are arrived at by linear interpolation, since the division of the sheet pile into increments virtually guarantees that the locations will not be directly analysed. One difficulty this comparison shows in the closed form solution's results for the support point (Location 3.) The discontinuity of moment and the limitations of linear interpolation are both apparent in the results at this point.

Another comparison is made with SPW911, a commercial sheet pile design program distributed by Pile Buck International. The results for the US and SI units cases can be seen from the program's graphical output in Figure 19 and Figure 20, respectively.

Anchored Sheet Pile Wall Analysis Using Fixed End Method Without Estimation of Point of Contraflexure

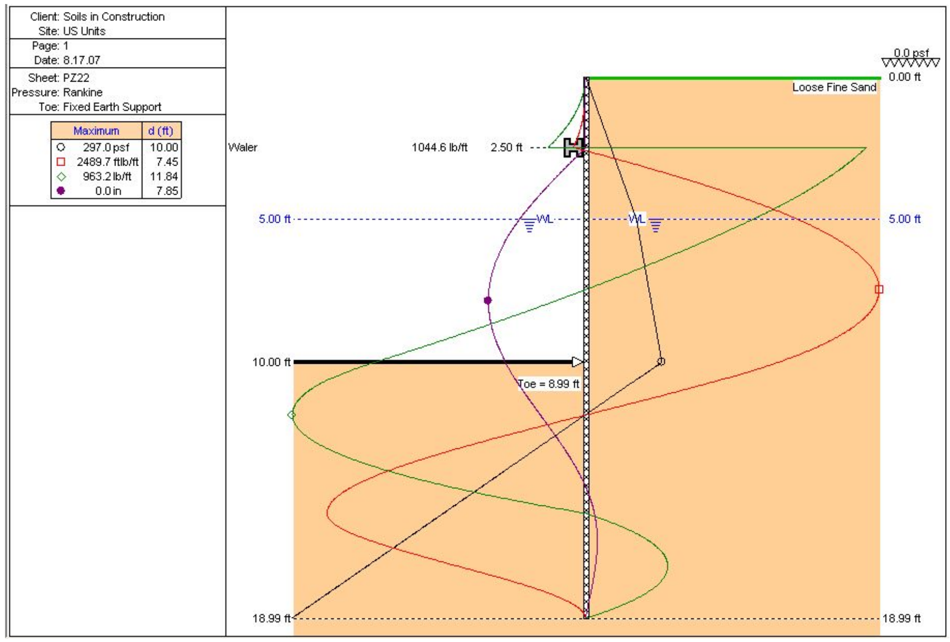

Figure 19 SPW 911 Results for US Units Case

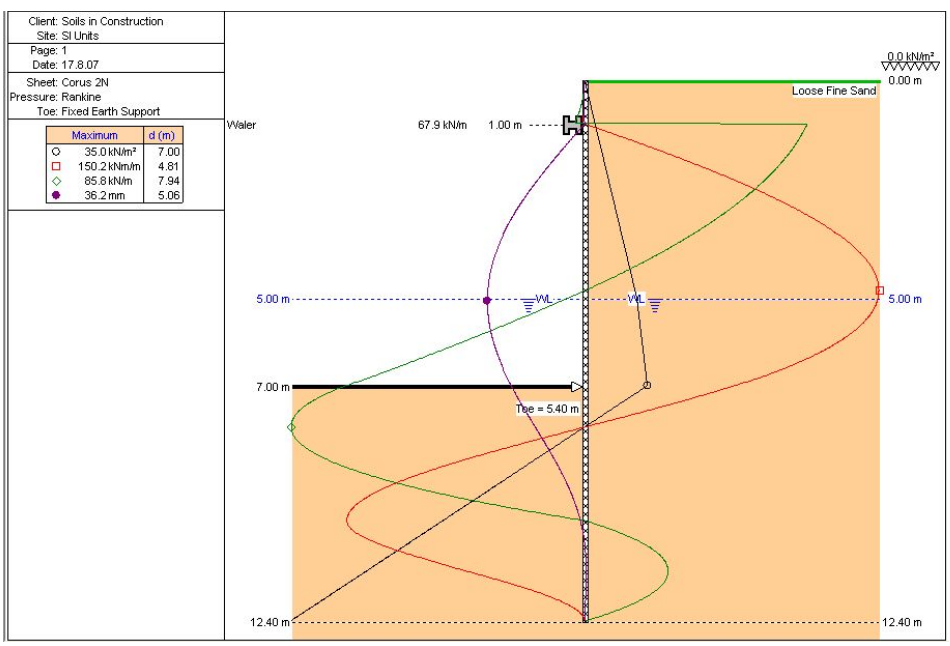

Figure 20 SPW 911 Results for SI Units Case

The results show that the SPW 911 are in general agreement with the closed form solution, although comparison of the deflection with the US units case is impossible because of excessive rounding in the SPW 911 results. The maximum shear returned by SPW 911 is lower than that of the closed form solution because the latter reports the shear at the unadjusted penetration of the sheeting, not further up as is done in SPW 911. It should also be noted that the adjusted penetrations recommended by the closed form solution are higher in both cases than SPW 911. SPW 911 consistently uses an adjustment factor of 1.2; if the unadjusted penetration from SPW 911 is back computed from the results, they are consistently higher than those of the closed form solution.

Discussion

The results from the analysis are in reasonable agreement both with a finite element solution (which one would expect) and with SPW 911. However, it should be noted that a) the number of test cases is very small and not statistically significant, and b) the scenario modelled by the closed form solution is very simple.

The "elastic line" solution shown here is shown to be practical, even with the limited computer power allocated to it. At one time such a solution was either too computationally intense or required some kind of graphical solution, but this is no longer the case. For fixed end sheet pile analysis, it is no longer necessary to assume a point of contraflexure using Blum's or any other type of analysis. Even with layered soils, it is reasonable to assume that an iterative solution could be developed to determine the unadjusted penetration of the sheeting, although care needs to be taken with the range of solutions for which this is practical. This limitation may be present in the presently used solutions but not apparent to someone who is not completely familiar with the background of the method being employed.

The one item that deserves the most attention for both the "elastic line" solution developed here and those solutions which assume a point of contraflexure concerns the adjustment applied to the penetration of the sheeting to compensate for the residual reaction at the toe. The fixed end method tends to be conservative to start with, and whatever factors of safety, load or resistance used need to take this into consideration versus those used in free end analysis. However, the fact that, for a wide variety of cases, strict application of Eq. 29 (or its equivalent for layered or cohesive soils) would result in differing results should give pause to the blind application of Eq. 30. Again the computation power available obviates the need to continue this assumption. In Figure 3, for the free end method is is recommended to increase the penetration by 20-40% (erroneously designated as a "factor of safety.") As this can be readily computed, this is further reason to abandon the use of a fixed extension of the sheet piling below the excavation line.

One thing that should be noted about the solutions described in this paper is that they are only performed for a very simple pile-soil system. More complex soil profiles will give different results. In the case of methods such as Blum's and that used by SPW 911 which divide the beam at an estimated point of contraflexure, complex soil profiles may also violate either the assumptions or the laboratory conditions under which the simplified methods were developed in the first place.

Conclusion

The fixed end method of anchored wall sheet pile analysis has been a viable method for a long time. The "elastic line" solution shown in this paper—albeit for a simple case—demonstrates that this method can be applied with the computational power available. The solution given in this paper results in both a series of charts (or tables) and a simple online routine that makes it ideal for academic use. Additionally, it was shown that using a fixed ratio of the unadjusted to the adjusted penetration could have unconservative results, but that this ratio as well can be computed without difficulty.

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