Pile-Driving Impact

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THIS CALCULATION, performed in the IBM Technical Computing Bureau, provided a complete numerical analysis of the behavior of a pile when struck by a pile-driving hammer. The results indicated what stresses occurred within the pile and capblock, as well as the penetration of the pile into the earth. This is an example of the replacement of costly experimentation by economical calculation.

Previously, using only key-driven calculators, it has been possible to study only partially a few isolated cases of the behavior of a pile under an impact. This was due to the high cost in time and money associated with each case studied. By employing the high speed and accuracy of IBM electronic calculating machines to perform this repeated formula evaluation, it is possible to study the behavior of numerous pile types. The cost of each case studied thereby will be substantially reduced.

When piles are driven as a foundation for a building or other structure, the load that they will carry safely usually is determined by measuring the penetration under the last blow of the hammer and substituting this figure in a formula. Many different formulas are in use, and they vary widely in the answers they give. This calculation provides a means of mathematically testing the accuracy of these formulas when applied to various types of piles and various ground conditions.

Furthermore, when the engineer has been asked what stresses the pile or the pile-driving equipment should be designed to withstand, he has often been at a loss for an answer. He has known only that certain practices and equipment have proved successful in the past. To do something new has meant proceeding by the often costly process of trial and error.

Because a pile is an object of considerable length, pile driving is a problem in longitudinal wave transmission and impact, the basic principles of which were first investigated 100 years ago by the French mathematicians, Saint Venant and Boussinesq. These principles have been ably set forth by L. H. Donnell in ASME Transactions APM-52-14. However, the problem is a very complicated one. The method of solution offered herein is based on approximate integration using a step-by-step calculation, and is of general interest because it can be applied to other impact problems. The calculation can be done by hand with a slide rule or desk calculating machine; however, modern electronic digital calculating machines are especially well adapted to make the calculation quickly and easily.

General Case—Basic Theory

For purposes of analysis, the hammer, pile, etc., will be represented by a series of concentrated weights separated by weightless springs. Subscripts $m-1$, m, $m+1$, etc., will be used to denote order of position, and superscripts $n-1$, n, $n+1$, etc., will be used to denote order of time.

Referring to Figure 1, let W_{m-1}, W_m and W_{m+1} represent successive weights, and S_{m-1}, S_m and S_{m+1} the springs underneath the respective weights. Let the initial positions

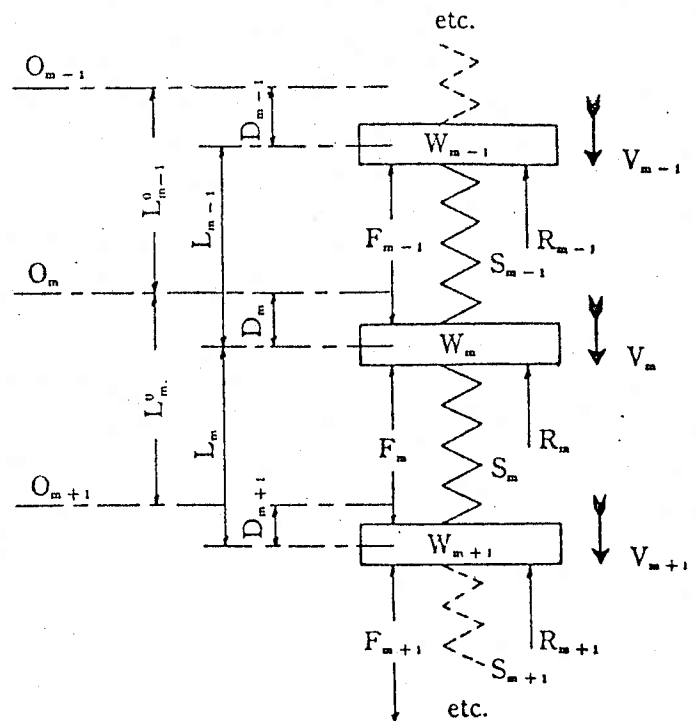FIGURE 1

of weights W_m, etc., at the beginning of impact be indicated by O_m, etc., and the initial lengths of the springs, by L_m^0, etc. Also, let

D_m, etc. = instantaneous displacements (inches).

L_m, etc. = instantaneous lengths (feet).

C_m, etc. = $12[L_m^0 - L_m] = D_m - D_{m+1}$ = instantaneous amount of spring compression (inches).

K_m, etc. = elastic constants for springs S_m, etc. = force required to produce $1''$ of compression C_m, etc. (pounds per inch).

F_m, etc. = instantaneous forces (pounds) resulting from C_m, etc.

V_m, etc. = instantaneous velocities of W_m, etc. (fps).

R_m, etc. = external forces, such as ground resistance, affecting the motion of W_m, etc. (pounds).

Z_m, etc. = instantaneous net force acting on W_m, etc. = $F_{m-1} - F_m - R_m$, etc. (pounds).

t = time (seconds).

Also, let superscripts $n-1$, n, $n+1$, etc., denote successive time intervals Δt, so small that with negligible error it may be assumed that all forces and velocities remain constant during each time interval. Then, if by previous calculation or otherwise, the values of V and D are known for some particular time interval $n-1$, all values for C, F, Z, V and D for the next time interval can be calculated as follows:

Let V_m^{n-1} and D_m^{n-1}, etc., represent definite known values time interval $n-1$, and let C_m^n, F_m^n, etc., represent definite values to be calculated for time interval n. Then the following general formulas apply:

$$C_m^n = D_m^{n-1} - D_{m+1}^{n-1} \tag{1}$$

$$F_m^n = K_m C_m^n \tag{2}$$

$$Z_m^n = F_{m-1}^n - F_m^n - R_m^n \tag{3}$$

$$V_m^n = V_m^{n-1} + \Delta V_m^n = V_m^{n-1} + \left(\frac{32.17\Delta t}{W_m}\right) Z_m^n \tag{4}$$

$$D_m^n = D_m^{n-1} + \Delta D_m^n = D_m^{n-1} + (12\Delta t) V_m^n \tag{5}$$

ΔV above is evaluated by using the standard formula for change of velocity

$$(v_1 - v_2) = \frac{\text{Force} \times \text{Time}}{\text{Mass}} = \frac{g\,Ft}{W},$$

where W is the weight, and g is the acceleration of gravity. ΔD above is evaluated by the formula for distance traveled $s = vt$ with the coefficient 12 introduced to convert to inches.

Careful consideration of the above formulas will disclose that the force at the beginning of an interval is used to calculate the velocity at the end of the interval, and then this end velocity is used to calculate the distance traveled during the same interval; therefore, the forces, velocities, and displacements used are slightly out of step with one another, depending on the size of the time interval.

These five formulas are used repeatedly until the calculation has been carried as far as necessary. They apply whether or not the successive weights, elastic constants, and external forces are equal or unequal. Furthermore, the values of K, R, and Δt may be changed from interval to interval according to any definite formula, or suddenly as required. Sudden changes are required, for instance, when the stress in a material reaches the yield point, when a weight loses contact with a spring designed only for compression, when a coefficient of restitution is introduced, or when a ground resistance force must be made negative so as to resist temporary upward movements. Such conditions are called *boundary conditions*. Some of the more complicated digital calculators can take care of these boundary conditions automatically.

Choice of Lengths L and Time Intervals Δt

It should be borne in mind that formulas (1) to (5) involve the use of small but finite increments, not infinitesimals. The lengths L and the time intervals Δt must, therefore, be chosen small enough to suit each particular type of problem. For each new type of problem halving or quartering the size of the units and recalculating part of the problem must be tried until it is found that the use of smaller units makes a negligible difference in the peak stresses that occur soon after impact. The time interval can be changed while a calculation is in progress by inserting a new value for Δt in formulas (4) and (5), although this change is subject to the limitation pointed out in Step 2 below.

Illustrative Problem

A pile of non-uniform section as shown in Figure 2 is to be driven through water or very soft mud to a hard layer of ground which is capable of resisting a maximum force of 600,000 pounds under the pile point. If the point of the pile starts to move upward momentarily, a negative frictional force of 100,000 pounds must be assumed, acting to hold the point down. No other side frictional forces are to be considered. The calculation is made to determine the final penetration per blow at which the assumed point resistance of 600,000 pounds will be developed.

Side friction along the pile has been omitted from this problem so as to allow the stress wave to travel with the known speed of stress (or sound) in the pile material and thus provide one way of checking the calculation. The method would apply equally well no matter what values were assigned to side friction.

Step 1. Decide on the time interval Δt. From previous experience a time interval of 1/4000 second has been chosen as being small enough to give accuracy within about 5%.

Step 2. Decide on lengths L. These must be at least as great as the distance stress will travel in the chosen time interval Δt; otherwise the stress wave will run ahead of the calculation, and the results will be meaningless. It is recommended that L be made equal to twice the distance that stress would travel in the chosen time interval. The upper part of this pile is entirely of steel, and the known speed of stress in steel is 16,800 fps; therefore, the recommended length for L is (16,800 × 2) ÷ 4000 = 8.4 feet. The pile length, plus a little added for the follower, happens to be a multiple of this figure; therefore, 8.4 feet can be used throughout. If an odd length were required, it would be inserted at the point of the pile.

Step 3. Prepare a diagram as per Figure 3 showing how the ram, capblock, follower, and pile are to be represented for purposes of calculation. The individual weights W_1, etc., are calculated so as to give a weight distribution closely equivalent to that of Figure 2.

Step 4. Prepare a tabulation of all constants required for formulas (1) to (5) as per Figure 4. This is readily done by considering the weight, cross-section, and modulus of elasticity of each portion of Figure 2 equivalent to a single spring or weight in Figure 3. The elastic constant K_1 for the wooden capblock must be determined by experiment or must be assumed. For this problem, a value of 6,400,000 pounds per inch has been assumed, which represents a rather stiff capblock, perfectly elastic.

Step 5. If the work is to be done by hand, it is conveniently tabulated as shown in Figure 5, which covers only the first three time intervals. In order to start the calculation, it is necessary to have a value for V and a value for D in the first time interval. The value of V, representing the velocity of the ram at the beginning of
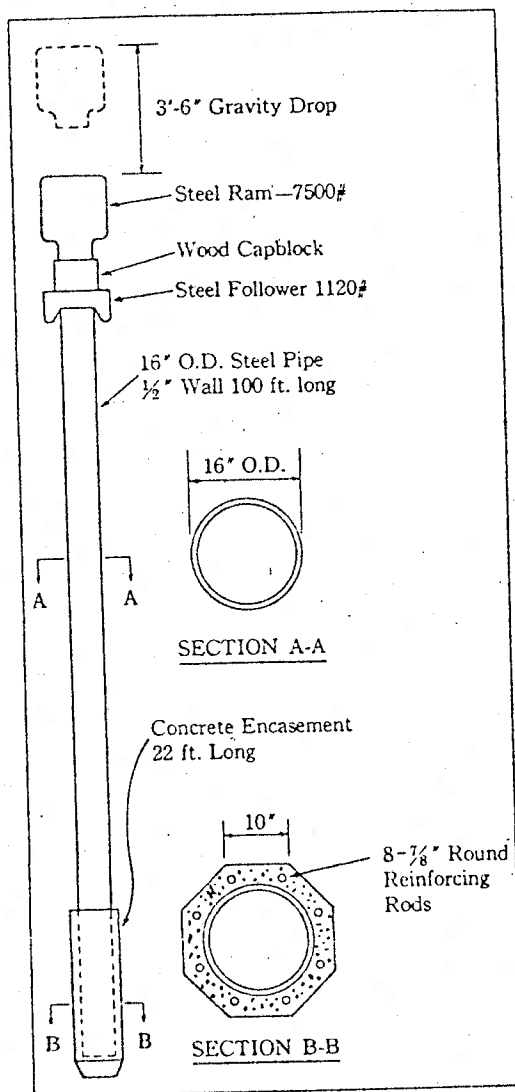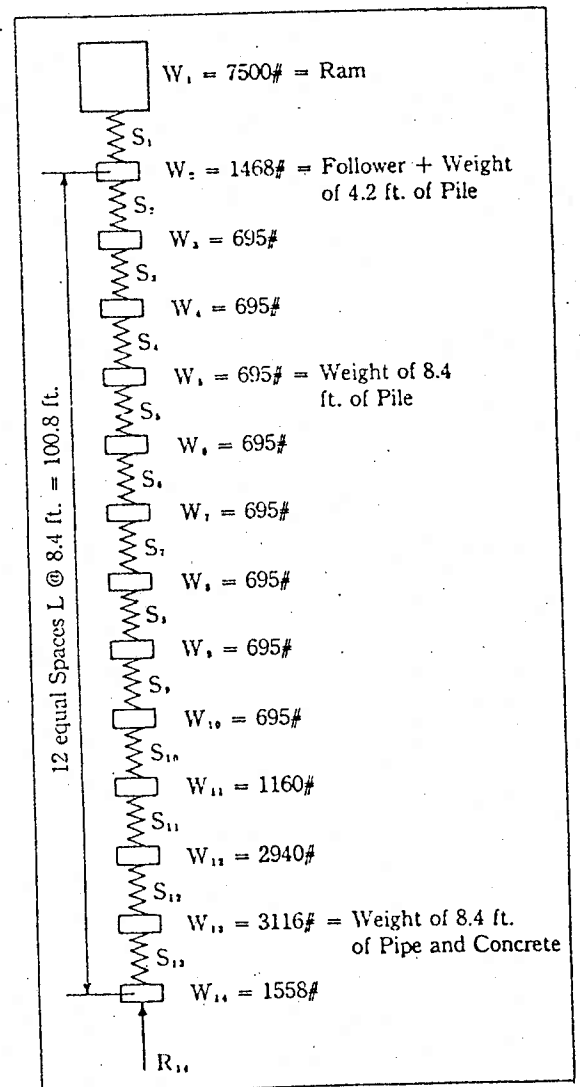

FIGURE 2

FIGURE 3

Subscript "m"	W	K	R	$\dfrac{32.17 \Delta t}{W}$	$12 \Delta t$
1	7500#	6,400,000#	0	1/932,500	0.003
2	1468	7,206,000	0	1/182,500	0.003
3	695	7,206,000	0	1/86,500	0.003
4	695	7,206,000	0	1/86,500	0.003
5	695	7,206,000	0	1/86,500	0.003
6	695	7,206,000	0	1/86,500	0.003
7	695	7,206,000	0	1/86,500	0.003
8	695	7,206,000	0	1/86,500	0.003
9	695	7,206,000	0	1/86,500	0.003
10	695	7,206,000	0	1/86,500	0.003
11	1160	14,900,000	0	1/145,000	0.003
12	2940	19,600,000	0	1/366,000	0.003
13	3116	19,600,000	0	1/388,000	0.003
14	1558		See Note	1/194,000	0.003

NOTE: R_{14} assumed $= F_{13}$ until F_{13} reaches 600,000#. Thereafter $R_{14} = 600,000$# if W_{14} is moving down and $-100,000$# if W_{14} is moving up.

FIGURE 4

n	C_1 $D_1^{n-1} - D_2^{n-1}$	F_1 $6,400,000 C_1^n$	Z_1 $-F_1^n$	V_1 $V_1^{n-1} + \dfrac{Z_1^n}{932,500}$	D_1 $D_1^{n-1} + .003 V_1^n$
	Inches	Pounds	Pounds	Ft. per Sec.	Inches
1	0	0	0	14.2500	.04275
				$-.2934$	.04187
2	.04275	273,600	$-273,600$	13.9566	.08462
	.08462 .00449			$-.5499$	.04022
3	.08013	512,783	$-512,783$	13.4067	.12484

n	C_2 $D_2^{n-1} - D_3^{n-1}$	F_2 $7,206,000 C_2^n$	Z_2 $F_1^n - F_2^n$	V_2 $V_2^{n-1} + \dfrac{Z_2^n}{182,500}$	D_2 $D_2^{n-1} + .003 V_2^n$
	Inches	Pounds	Pounds	Ft. per Sec.	Inches
1	0	0	0	0	0
2	0	0	273,600	1.4992	.00449
			512,783 32,409	2.6322	.01240
3	.00449	32,409	480,374	4.1314	.01689

NOTE: Line 3 is not complete because the Force $F_2^3 = 32,409$ can be used to calculate values for Z_3^3, V_3^3 and D_3^3 which are not shown. Velocity of Ram at Impact $= 14.250$ fps.

FIGURE 5

the impact, must be calculated by considering the distance it falls and allowing for hammer efficiency. For this problem, efficiency was assumed to be 90%, which gave a velocity of 14.25 fps to be used in starting the step-by-step calculation. The numerical value of the displacement D in the first time interval is obtained from the assumption that the ram continues to move with undiminished velocity through the first 1/4000 second after the impact. For an impact velocity of 14.25 fps this gives a displacement in the first time interval of 0.04275 inches. This

displacement then is used to calculate force F_1 for the second time interval, and so on. As the stress wave travels down the pile, additional columns are needed in groups of five, all headed by the basic formulas (1) to (5) using constants taken from Figure 4.

If the work is done by an electronic digital calculator, the results will be tabulated by the machine as shown in Figure 6. This is a condensed tabulation which shows time intervals 1 to 5 and some of the later time intervals. For the later time intervals, only the data for the top

Time Interval "n"	Subscript "m"	Forces "F" pounds	Forces "R" pounds	Velocities "V" fps	Displacements "D" inches
1	1	0	0	14.2500	0.04275
2	1	273,600	0	13.9566	0.08462
	2	0	0	1.4992	0.00450
3	1	512,783	0	13.4067	0.12484
	2	32,409	0	4.1314	0.01689
	3	0	0	.3747	0.00112
4	1	690,869	0	12.6658	0.16284
	2	113,621	0	7.2944	0.03877
	3	8,100	0	1.5946	0.00591
	4	0	0	.0937	0.00028
5	1	794,001	0	11.8144	0.19828
	2	236,839	0	10.3473	0.06982
	3	40,547	0	3.8638	0.01750
	4	2,024	0	.5390	0.00190
	5	0	0	.0234	0.00007
20	1	238,945	0	5.7933	0.54314
	2	223,200	0	5.2403	0.50414
	13	522	0	0.0199	0.00009
	14	0	522	0	0
28	1	166,719	0	3.9070	0.65498
	2	221,163	0	4.6805	0.63125
	13	366,135	0	3.5019	0.02919
	14	0	366,135	0	0
30	1	139,864	0	3.5942	0.6770
	2	195,695	0	4.0635	0.6566
	13	834,305	0	5.1612	0.0581
	14	0	600,000	1.2734	0.0038
46	1	465,569	0	0.3514	0.78710
	2	539,162	0	3.7305	0.70211
	13	424,886	0	3.8138	0.26202
	14	0	600,000	0.0933	0.22918
56	1	182,869	0	−5.0749	0.69620
	2	365,779	0	−0.6060	0.68104
	13	334,104	0	0.4408	0.30442
	14	0	600,000	−0.9241	0.28328
57	1	97,066	0	−5.1790	0.68067
	2	280,059	0	−1.6087	0.67621
	13	414,362	0	−0.2967	0.30353
	14	0	−100,000	1.8713	0.28890
58	1	28,516	0	−5.2096	0.66504
	2	157,825	0	−2.3172	0.66926
	13	286,881	0	−0.5238	0.30196
	14	0	600,000	0.1696	0.28940
59	1	−27,017	0	−7.4977	0.64254
	2	15,191	0	−2.5485	0.66161
	13	246,113	0	−0.4272	0.30068
	14	0	600,000	−1.7537	0.28414

FIGURE 6

and bottom of the pile are included, because these are the controlling factors in the calculation.

The calculation begins in the first time interval with a ~~m~~ velocity of 14.25 fps and a displacement of 0.04275 ~~...~~les. The stress wave travels down the pile, bringing each successive section of the pile into action. For example, in the fourth time interval, the velocity V_3 amounts to only 1.5946 fps with a displacement of D_3 of only 0.00591 inches, and a total force F_3 of only 8100 pounds.

It will be observed that, to represent the specified ground resistance, R_{14} is given a value equal to F_{13} until F_{13} exceeds 600,000 pounds, as it does first at time interval 30. From this point on, R_{14} remains at 600,000 pounds, except that when V_{14} becomes negative, as it does in time interval

56, R_{14} is given a value of $-100,000$ pounds in the next time interval, which, in this case, is interval 57. In interval 57, the velocity V_{14} is again positive; therefore, R_{14} is again given a value of 600,000 pounds in interval 58, and so on. In the meantime, the values of the displacements D_{14} have increased gradually to a maximum of 0.28940 in interval 58, after which they remain practically unchanged. It will also be observed that the capblock force F_1 becomes negative in time interval 59. This means that the hammer ram has separated from the capblock; therefore, at this point, the hammer ram is dropped from the calculation.

Figure 7 is a graphic representation of the results of the calculation. A separate curve is plotted for each weight W,
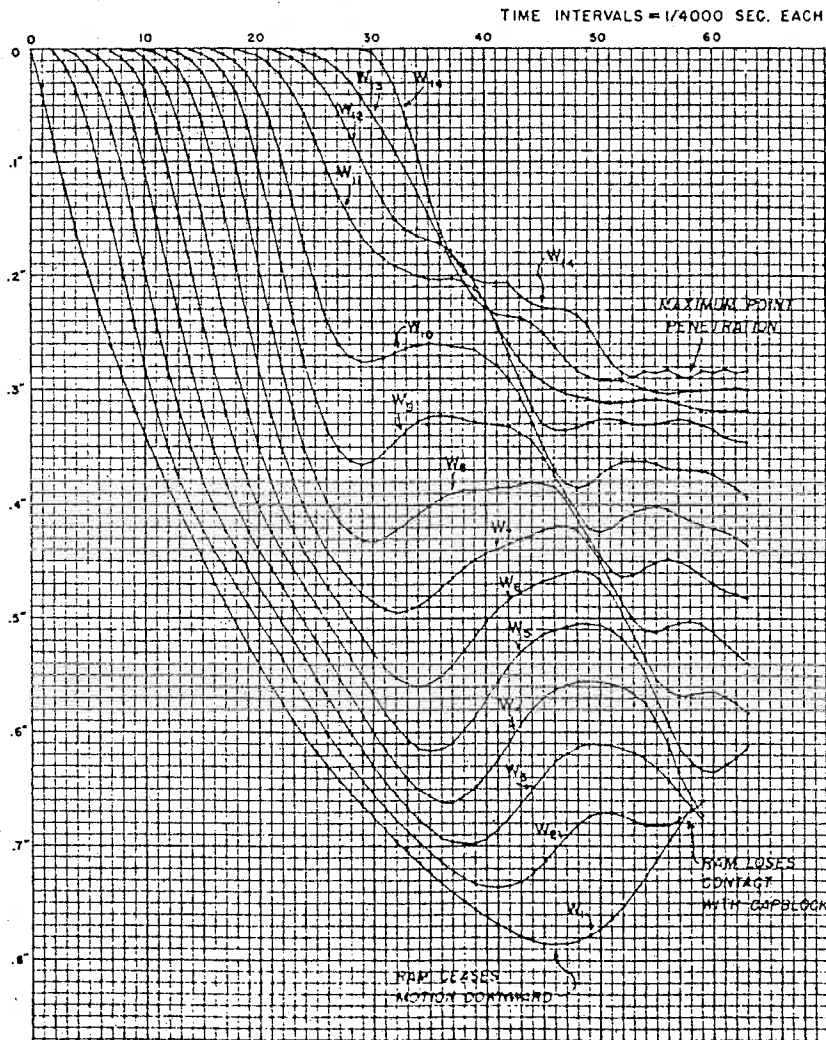

FIGURE 7. DISPLACEMENTS

and the curves represent the relationship between displacement and time. It can be seen from the curve for W_1 that the hammer ram curve crosses the pile head curve in the 58th time interval. On the curve for W_{14}, it can be seen that the penetration reaches a maximum in the 53rd time interval and then fluctuates slightly in the succeeding time intervals, reaching practically the same maximum again in interval 58. The calculation should always be carried beyond the first maximum of penetration in order to make sure that only slight fluctuations in penetration will occur thereafter.

Checking the Calculations

The total energy of the system for any particular time interval can be obtained by adding the kinetic energies of the individual weights, the potential energies of the individual springs, and the total work performed in overcoming the various external forces. The total should equal the energy of the ram just before impact.

The total momentum of the system for any particular time interval can be obtained by adding the products of each mass multiplied by its instantaneous velocity and the products of each external force multiplied by the total time it has acted. The total should equal the momentum of the ram just before impact.

Neither check is exact because of minor inaccuracies in this method, but a sudden variation between one time interval and the next indicates a numerical error. If the total varies by more than about 5%, consideration should be given to reducing the time interval and possibly the lengths L. If the work is done by hand, it is recommended that checks be made at every 10th interval. If the work is done by an automatic calculator, it may be possible to include a running check as part of the setup. The energy check is to be preferred to the momentum check as it is more complete. Plotting the calculated results for displacements D as per Figure 7 is also an excellent check on the reasonableness of the results. Curves for separate calculations may be readily compared.

Recalculation for Change in Ground Resistance

A change in the resistance near the point of the pile will change only a half or a third of the total calculation. Piles may, therefore, be recalculated for various point resistances with a considerable saving of effort as compared with an entirely new calculation.

DISCUSSION

Mr. Sheldon: I would like to make a few comments. I have been much impressed with Mr. Smith's handling and understanding of the phenomena which go into these piles. He has not employed calculus, but he has really derived for you the partial differential equation for the motion of the pile with boundary conditions.

The equation for the displacement $D(x,t)$ in a one-dimensional, inhomogeneous elastic medium is:

$$\rho(x)\frac{\partial^2 D}{\partial t^2} = \frac{\partial}{\partial x}\left\{ Y(x)\frac{\partial D}{\partial x}\right\} + f(x,t)$$

where $\rho(x)$ is the mass density, $Y(x)$ is Young's modulus, and $f(x,t)$ is the applied stress. The simplest difference system by which we can replace the above differential equation is, in the notation of Mr. Smith,

$$\rho_m \frac{D_m^{n+1} - 2D_m^n + D_m^{n-1}}{(\Delta t)^2} =$$
$$\frac{Y_{m+1}[D_{m+1}^n - D_m^n] - Y_m[D_m^n - D_{m-1}^n]}{(\Delta x)^2} + f_m^n.$$

The solution of this difference system is exactly equivalent to the solution of the difference system derived by Mr. Smith from first principals. To see this, note that

$$K_m = \frac{Y_m}{\Delta x}, F_m^n = K_m[D_m^n - D_{m-1}^n], R_m^n = f_m^n \Delta x,$$
$$\frac{W}{32.17} = \rho_m \Delta x \quad \text{and} \quad V_m^n = \frac{D_m^n - D_m^{n-1}}{12\Delta t}.$$

Mr. Smith chose an interval $\Delta t = \Delta x/2c$ ($c =$ sound speed), so that he had a safety factor of 2 in the Courant condition for the stability of numerical integration of hyperbolic type equations.

We solved this problem on the card-programmed calculator at the technical computing bureau. We chose the CPC for solution because Mr. Smith had quite complicated boundary conditions imposed. For one thing, the resistance of the ground is a non-linear function. It is 600,000 pounds upwards when the last weight is moving down and 100,000 pounds downward when the last weight is moving up. Another condition that has to be provided for is that the capblock and follower are not attached to the pile itself, so that after a certain period of time the capblock and follower fly off the top of the pile. Using the card-programmed calculator with its facility to list answers as we go along, we were able to observe the sign of the velocity at the bottom of the pile and also whether there was a tension or compression in the capblock. As soon as the condition which bounded this motion changed, we were able to insert a new instruction card which would take care of the new condition. This is a much simpler procedure than attempting to put these conditions into the control panel of a machine. We have solved, totally, eight cases of this pile-driving work, and for the last six cases there was an additional complication in the auxiliary conditions. Mr. Smith decided to take account of the fact that the capblock was made of wood and, therefore, was not perfectly elastic. We changed the elastic constant of the capblock according to whether the capblock was being compressed or was expanding.

The problem runs at about one step in the time every three minutes. This is an average figure, taking account of changing the program cards to take care of auxiliary conditions.

Dr. Aronofsky: I do not understand the boundary conditions at the top. The weight of the ram is 7,500 pounds. Is there any condition imposed?

Mr. Sheldon: The weight at the top is a freely falling weight; so the boundary condition at the upper end of the pile is that at $t = 0$ the ram has a certain definite velocity, and all the other weights are not moving.

Dr. Aronofsky: Is there any assumption about resistance along the lateral side of the pile all the way down?

Mr. Sheldon: In one case there was just the resistance at the bottom. In another case the resistance was applied in the middle.

Mr. Smith: In Figure 4 the only resistance that is inserted is the last one. All the other resistances are zero.

However, if desired, you can put in as many resistances as there are weights.

Dr. Buchholz: I think such studies have been made on analog equipment. You replace this type of system by a network of little capacities and provide certain nonlinear elements to take care of boundary conditions and special conditions. You run into a bit of a problem in the case of the capblock leaving the rest of the system. I don't know whether this problem has been done, but I imagine it might be possible to do so.

Mr. Moncreiff: Was special wiring used for this problem?

Mr. Sheldon: No effort was made to change the standard setup at all. We made it very simple so that it took about one day to plan for the machine.

Mr. Moncreiff: The calculation is simple enough so that you could save time by wiring a special control panel.

Mr. Sheldon: That is true.