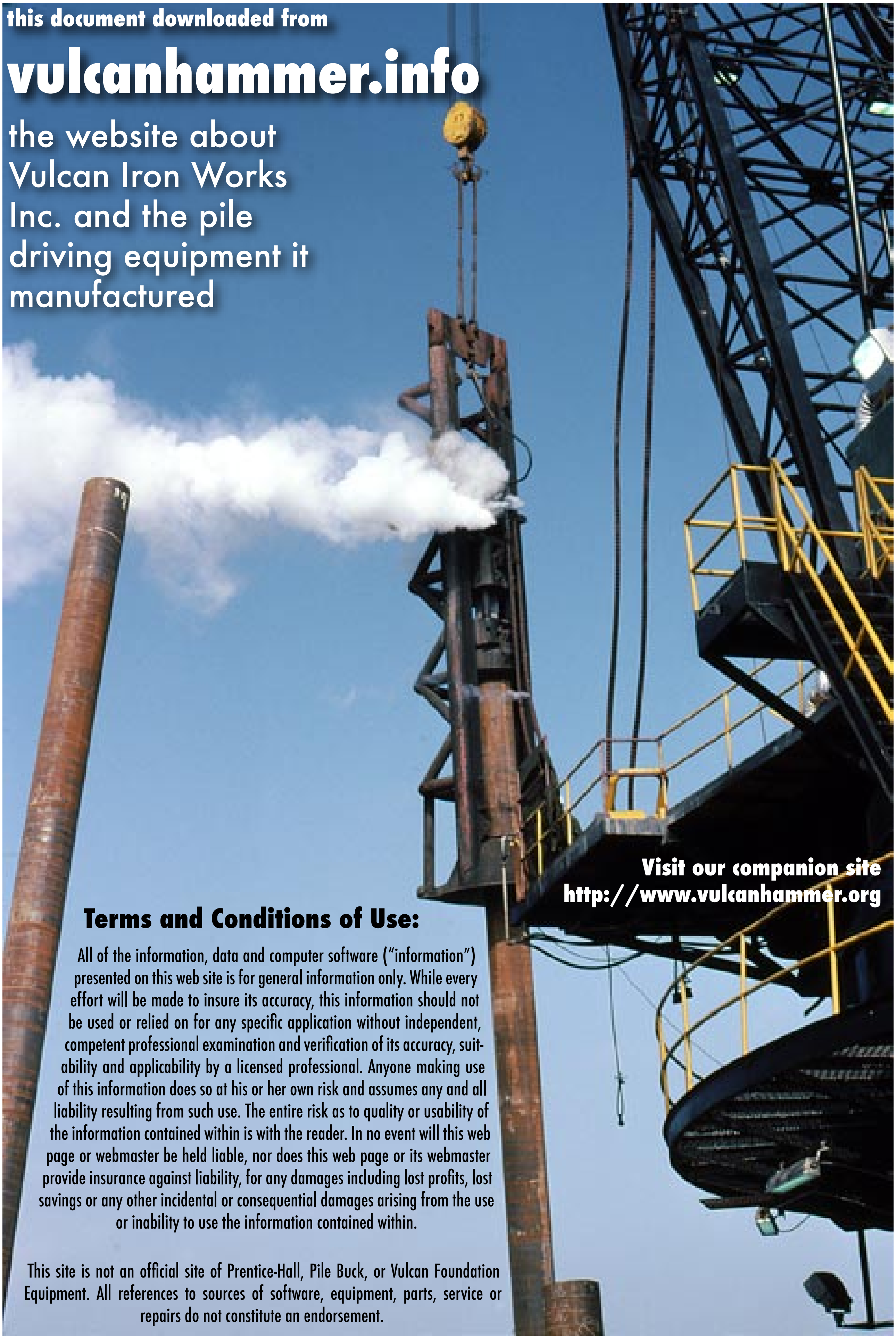

Impact and Longitudinal Wave Transmission

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The necessary formulas for a numerical method of calculation are derived without the use of calculus or other advanced mathematics; an illustrative problem is solved in complete detail; eight different types of impact are discussed; methods are given for taking account of friction, damping, coefficient of restitution, plastic flow, and gravity; branched systems are discussed briefly; methods are given for checking the calculated results.

Introduction

The purpose of this paper is to present, as simply as possible, a numerical method that may be used with slide rule, desk calculator, or electronic digital calculator for approximate solution of problems such as the action of a forging hammer, a [illegible] driver, or a pile-driving hammer and pile. The method [illegible] be applied to problems of longitudinal vibration without [illegible] and is especially suited to those where many degrees of freedom, irregular weight distribution, or other complications are involved. [illegible] applying to such [illegible] has been discussed by [illegible],[2] Donnell (2), and others.

The following notation will be used:

[illegible] $m-1$, m, etc., are numbers designating particular weights and associated springs, and may be used as subscripts.
1, 2, 3, … $n-1$, n, etc., are numbers designating time intervals. [illegible] used similarly to designate the initial instant.

The following apply to any time interval n:

D_m = displacement of weight m measured from its initial position, in.
C_m = compression of spring m, in.
F_m = force exerted by spring m, lb
Z_m = net force acting on weight m, lb
V_m = velocity of weight m, fps

The following apply to the preceding time interval $n-1$:

d_m = displacement of weight m measured from its initial position, in.
c_m = compression of spring m, in.
v_m = velocity of weight m, fps

The following are usually constants:

W_m = magnitude of weight m, lb
K_m = spring constant for spring m, lb/in.
R_m = external force or resistance acting on weight m, lb
g = acceleration due to gravity (32.17 fps^2)
E = Young's modulus of elasticity, psi
A = cross-sectional area, sq in.
L = unit length, in.
Δt = time interval used for numerical calculation, sec
T_m = critical time interval for spring m, sec
T_{min} = minimum value of T_m, sec
ϕ = over-all factor of safety = $T_{min}/\Delta t$
ϕ_m = individual factor of safety for spring m = $T_m/\Delta t$

A number of other symbols used for special purposes only are defined as they occur.

The object or objects taking part in the action will be represented as a series of concentrated weights separated by weightless springs as shown in Fig. 1.

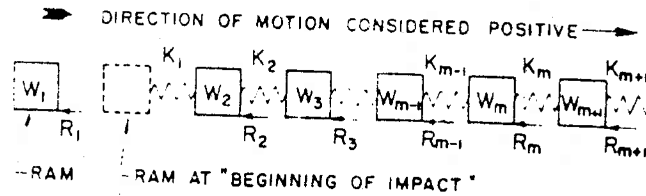

Fig. 1

The time during which the action occurs will be divided into small time intervals Δt such as 1/4000 sec, the size of the intervals being chosen to suit the nature of the problem. The intervals must be small enough so that with negligible error it may be assumed that all velocities, forces, and displacements will have fixed values during any particular interval.

The numerical calculation will be a step-by-step process in which the five variables D_m, C_m, F_m, Z_m, and V_m will be calculated for each weight or spring in each successive interval. It is therefore necessary to develop formulas for these five variables.

Development of Basic Formulas

D_m equals d_m plus an increment of displacement acquired during a single interval Δt. This increment may be evaluated as $v_m 12\Delta t$, based on the law that distance traveled = velocity × time. The coefficient 12 is required because D_m and d_m are expressed in inches and v_m in feet per second. The required formula for D_m is then

$$D_m = d_m + v_m 12\Delta t \qquad [1]$$

In Equation [1], the values of d_m, v_m, and Δt are always known, because Δt is the chosen time interval, and d_m and v_m are either given quantities or they have been calculated previously.

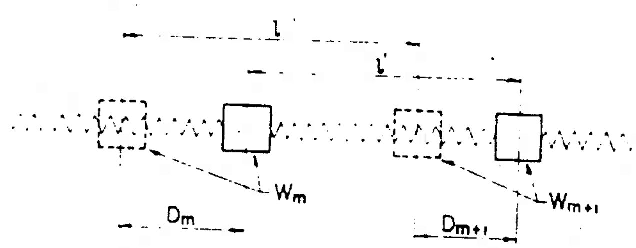

Fig. 2

To obtain a formula for C_m let the dotted squares in Fig. 2 represent the initial positions of weights m and $m+1$, and let the solid squares represent their positions in interval n. Then l will be the initial length of spring m and l' will be its length in interval n; also D_m and D_{m+1} will be the displacement of weights

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[2] Numbers in parentheses refer to Bibliography at end of paper.

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TABLE 2 NUMERICAL CALCULATION

n	D_1 $d_1+.004v_1$	C_1 D_1-D_2	F_1 $4500C_1$	Z_1 $-F_1$	V_1 $v_1+\frac{Z_1}{466.3}$	D_2 $d_2+.004v_2$	C_2 D_2-D_3	F_2 $5500C_2$	Z_2 F_1-F_2-100	V_2 $v_2+\frac{Z_2}{93.25}$	D_3 $d_3+.004v_3$	C_3 D_3	F_3 $6500C_3$	Z_3 F_2-F_3-10	V_3 $v_3+\frac{Z_3}{186.5}$
0	0 +.04	0 .04 −0	0	0	10.0000 −.386	0	0	0	0 180 −100	0 +.8578	0	0	0	0	0
1	0.04000 +.03846	0.04000 .07846 −.00343	180.00	−180.00	9.6140 −.7241	0 +.00343	0 .00343 −0	0	80.00 337.63 −18.86 −100	0.8578 +2.3458	0	0	0	0 18.86 −10	0
2	0.07846 +.03555	0.07503 .11401 −.01624	337.63	−337.63	8.8899 −.9435	0.00343 +.01281	0.00343 .01624 −.00019	18.86	218.77 439.96 −88.28 −100.00	3.2036 +2.6987	0 +.00019	0	0	8.86 88.28 [illegible] −10	0.01[illegible] [illegible]
3	0.11401	0.09777	439.96	−439.96	7.9464	0.01624	0.01605	88.28	251.68	5.9023	0.00019	0.00019	1.24	77.04	0.4606
4	0.14579	0.10594	476.73	−476.73	6.9241	0.03985	0.03782	208.01	168.72	7.7114	0.00203	0.00203	13.19	184.82	1.1516
5	0.17349	0.10280	462.60	−462.60	5.9320	0.07069	0.06285	345.67	16.93	7.8929	0.00784	0.00784	50.96	284.71	2.9782
6	0.19722	0.09496	427.32	−427.32	5.0156	0.10226	0.08251	453.80	−126.48	6.5367	0.01975	0.01975	128.37	315.43	4.6695
7	0.21728	0.08888	399.96	−399.96	4.1579	0.12840	0.08997	494.83	−194.87	4.4472	0.03843	0.03843	249.79	235.04	5.9298
8	0.23391	0.08772	394.74	−394.74	3.3114	0.14619	0.08404	462.22	−167.48	2.6514	0.06215	0.06215	403.97	48.25	6.1885
9	0.24716	0.09036	406.62	−406.62	2.4394	0.15680	0.06990	384.45	−77.83	1.8168	0.08690	0.08690	564.85	−190.40	5.1676
10	0.25692	0.09291	418.09	−418.09	1.5428	0.16401	0.05644	310.42	7.67	1.8990	0.10757	0.10757	699.20	−398.78	3.0294
11	0.26309	0.09148	411.66	−411.66	0.6600	0.17161	0.05192	285.56	26.10	2.1789	0.11969	0.11969	777.98	−502.42	0.3355
12	0.26573	0.08541	384.34	−384.34	−0.1642	0.18032	0.05929	326.09	−41.75	1.7312	0.12103	0.12103	786.69	−470.60	−2.1878
13	0.26507	0.07783	350.23	−350.23	−0.9153	0.18724	0.07496	412.28	−162.05	−0.0064	0.11228	0.11228	729.82	F_2-F_3+10 −307.54	−3.8368
14	0.26141	0.07420	333.90	−333.90	−1.6314	0.18721	0.09028	496.54	F_1-F_2+100 −62.64	−0.6781	0.09693	0.09693	630.04	−123.50	−4.4990
15	0.25488	0.07038	316.71	−316.71	−2.3106	0.18450	0.10557	580.63	−163.92	−2.4358	0.07893	0.07893	513.04	77.59	−4.0830
16	0.24564	0.07088	318.96	−318.96	−2.9946	0.17476	0.11216	616.88	−197.92	−4.5580	0.06260	0.06260	406.90	219.98	−2.9035
17	0.23366	0.07713	347.08	−347.08	−3.7389	0.15653	0.10554	580.47	−133.39	−5.9883	0.05099	0.05099	331.43	259.04	−1.5145
18	0.21870	0.08612	387.54	−387.54	−4.5700	0.13258	0.08765	482.07	5.47	−5.9296	0.04493	0.04493	292.04	200.03	−0.4420
19	0.20042	0.09156	412.02	−412.02	−5.4536	0.10886	0.06570	361.35	150.67	−4.3140	0.04316	0.04316	280.54	90.81	0.0449
20	0.17860	0.08700	391.50	−391.50	−6.2932	0.09160	0.04826	265.43	226.07	−1.8899	0.04334	0.04334	281.71	F_2-F_3-10 −26.28	−0.0960
21	0.15343	0.06939	312.25	−312.25	−6.9628	0.08404	0.04108	225.94	186.31	0.1078	0.04296	0.04296	279.24	F_2-F_3+10 −43.30	0.3282
22	0.12558	0.04111	185.00	−185.00	−7.3595	0.08447	0.04282	235.51	F_1-F_2-100 −150.51	−1.5061	0.04165	0.04165	270.72	−25.21	−0.4634
23	0.09614	0.01770	79.65	−79.65	−7.5303	0.07844	0.03864	212.52	F_1-F_2+100 −32.87	−1.8585	0.03980	0.03980	258.70	−36.18	−0.6574
24	0.06602	−0.00499	0	0	−7.5303	0.07101	0.03384	186.12	−86.12	−2.7819	0.03717	0.03717	241.60	−45.48	−0.9013
25	0.03590	−	0	0	−7.5303	0.05988	0.02631	144.70	−44.70	−3.2612	0.03357	0.03357	218.20	−63.50	−1.2418
26	0.00578	−	0	0	−7.5303	0.04684	0.01824	100.32	−0.32	−3.2646	0.02860	0.02860	185.90	−75.58	−1.6470
27	−0.02434	−	0	0	−7.5303	0.03378	0.01177	64.73	35.27	−2.8864	0.02201	0.02201	143.06	−68.33	−2.0134
28	−0.05446	−	0	0	−7.5303	0.02223	0.00828	45.54	54.46	−2.3024	0.01395	0.01395	90.67	−35.13	−2.2018
29	−0.08458	−	0	0	−7.5303	0.01302	0.00788	43.34	56.66	−1.6948	0.00514	0.00514	33.41	19.93	−2.0949
30	−1.11470	−	0	0	−7.5303	0.00624	0.00948	52.14	47.86	−1.1816	−0.00324	−0.00324	−21.06	83.20	−1.6488

... interval n. Then $C_m = l - l'$; but $l + D_{m+1} =$ ... it follows at once that

$$C_m = D_m - D_{m+1} \ldots \quad [2]$$

... directly proportional to C_m and to the spring ...

$$F_m = C_m K_m \ldots \quad [3]$$

... It will be noted that weight m is acted on by ... and by external force or resistance R_m; ... accelerating force Z_m acting on weight m is

$$Z_m = F_{m-1} - F_m - R_m \ldots \quad [4]$$

... an increment of velocity acquired during a ... This increment may be evaluated as

$$Z_m(\Delta t g/W_m)$$

... that change of velocity

= force × time ÷ mass

... for V_m then becomes

$$V_m = v_m + Z_m(\Delta t g/W_m) \ldots \quad [5]$$

Illustrative Problem

... problem shows how the foregoing formulas ...

... weighing 5 lb and moving at 10 fps strikes a ... stationary weights, each with a frictional re... weightless springs, as shown in Fig. 3. Calcu... motions and forces using a time interval of ...

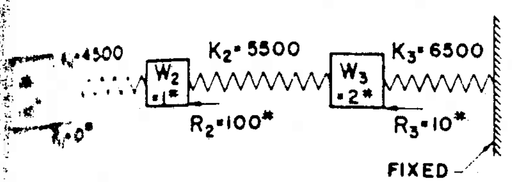


Fig. 3

... of constants, Table 1, used in the five for... the given problem.

Table 1 Tabulation of Constants

...	K_m	R_m, lb	$12\Delta t$	$\Delta tg/W_m$
...	...00	0	0.004	1/466.3
...	...500	100	0.004	1/93.25
...	...500	10	0.004	1/186.5

... calculation may then be performed and tabu... in Table 2.

... Table 2

... between lines for making computations have been ... interval (or line) 3. Such spaces are convenient ... done by slide rule.

... the values are computed for D_1, D_2, and D_3 first, ... C_2 and C_3 next, etc.

... R_2 and R_3 are frictional resistances they cannot be ... produce motion at the beginning of the calculation. ... shows up in Table 2 as follows:

... Z_2 and Z_3 = 0 (not —100 for Z_2 or —10 for Z_3)

... Z_3 = 0 (not —10)

4 Similarly, R_2 and R_3 act only to oppose motion; therefore when V_3 becomes negative on line 12, R_3 becomes positive starting with line 13. Similar reversals of sign occur on lines 14, 20, 21, 22, and 23.

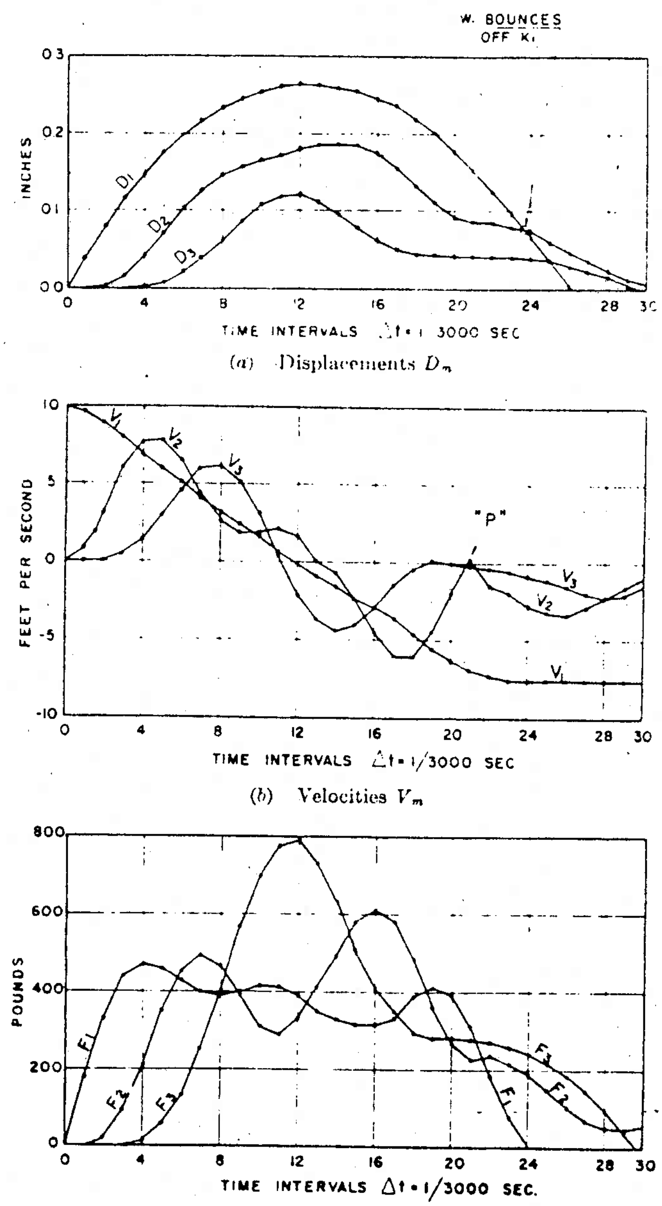
5 Spring K_1 is not designed to take tension; therefore when C_1 becomes negative on line 24 it means that K_1 has ceased to act and W_1 has bounced off with a velocity of —7.5303 fps.

6 Impact ends in interval 24; therefore the duration of impact is approximately $^{24}/_{3000}$ or 0.008 sec.

7 The numerical calculation may be started at the beginning of impact or at any other instant for which conditions are given. For instance, Table 2 might have been started at line 6 if the given conditions had been as follows:

D_1 = 0.19722	V_1 = 5.0156
D_2 = 0.10226	V_2 = 6.5367
D_3 = 0.01975	V_3 = 4.6695

8 The numerical calculation may start from known forces. For instance, if the values of F_1 shown in Table 2 had all been



(a) Displacements D_m

(b) Velocities V_m

(c) Forces F_m

Fig. 4 Plotted Values From Table 2

known or assumed, the action of W_2, K_2, W_3, and K_3 could have been calculated exactly as in Table 2 except that columns D_1, C_1, Z_1 and V_1 could be omitted.

9 It follows from items 7 and 8 that certain problems may be solved that involve longitudinal wave transmission but do not involve impact.

10 Some electronic digital calculators can be programmed to handle automatically such special conditions as 3, 4, and 5.

11 Problems involving a larger number of weights and springs can be handled in exactly the same way, except that more lines would appear in Table 1 and more columns (and also probably more lines) in Table 2.

12 Ordinarily, Table 1 would be prepared by an engineer or mathematician, but the numerical calculation of Table 2 may be performed by others.

Plotting. Plotting the results of the numerical calculation is an excellent method of checking accuracy. The values of D_m, V_m, and F_m should plot as "smooth" or "stable" curves. Any sharp peak, or any sharp "wiggle" in any of the curves indicates either that a numerical error has been made or that the time interval used was too large.

The values listed in Table 2 have been plotted in Fig. 4. The curves are all reasonably smooth, except for the sharp peak that is marked P in Fig. 4(*b*). It may be concluded that the numerical calculation was reasonably accurate up to and including interval 19 but that thereafter something occurred that produced definite [illegible]. A check of the numerical work has disclosed no [illegible] therefore the results of the calculation up to interval 19 [illegible] taken as correct, but the calculation should be repeated from this point on, using a smaller time interval.

Weight Distribution. It is recommended that the weight of a long rod or similar object be distributed to the ends of each unit length, not toward the center. For instance, a uniform rod to be [illegible] six weights would be divided into five unit lengths. [illegible] length weighs 10 lb the distribution would be made as [illegible] Fig. 5.

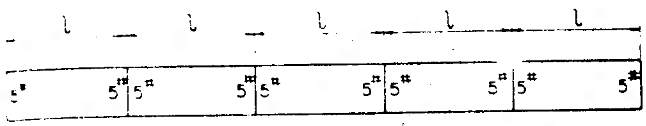


Fig. 5

[illegible] purposes of calculation this would result in the diagram of Fig. 6.

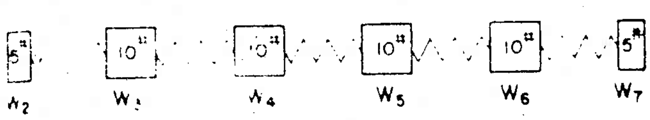


Fig. 6

[illegible] uniform, it is recommended that the [illegible] in proportion to the location of the center [illegible] as in Fig. 7.

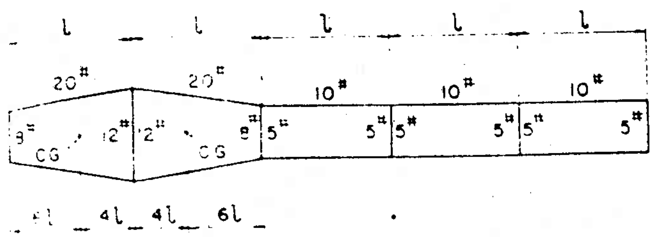


Fig. 7

[illegible] reserved for the [illegible]

The resulting diagram for purposes of computation is shown in Fig. 8.

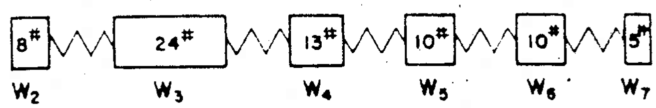


Fig. 8

If the object is not continuous but consists of individual sections of considerable weight with intermediate springs, such as a railroad train, it ordinarily may be represented with sufficient accuracy as shown in Fig. 9 in which W_1 is the weight of the

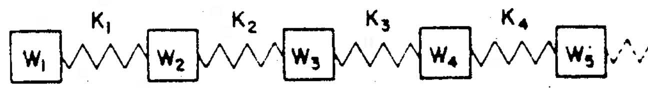


Fig. 9

locomotive, and W_2, etc., are the weights of the individual cars. Any elasticity in the structure of the car itself may be added to the spring constants of the couplers at either end using Formula [8] given later. If extreme accuracy is required, each weight as noted may be divided into two or more weights with intermediate springs by following the method of Figs. 5 and 6.

Spring Constant K_m. If spring K_m represents the elasticity of a unit length l of a rod of uniform section the formula for K_m is

$$K_m = AE/l \qquad [6]$$

If K_m consists of two or more springs in parallel its value is obtained by adding the spring constants of the individual springs, thus

$$K_m = k_1 + k_2 + k_3, \text{ etc.} \qquad [7]$$

If K_m consists of two or more springs in series its value is obtained by adding reciprocals, thus

$$1/K_m = 1/k_1 + 1/k_2 + 1/k_3, \text{ etc.} \qquad [8]$$

If K_m represents a uniformly tapered square section with sides at one end equal to α and at the other end equal to β, then

$$K_m = E\alpha\beta/l \qquad [9]$$

If K_m represents a uniformly tapered round section of diameter α' at one end and β' at the other, then

$$K_m = \frac{\pi}{4} E\alpha'\beta'/l \qquad [10]$$

If K_m represents a uniformly tapered rectangular section having sides a' and b' at one end and corresponding sides A' and B' at the other end, then

$$\text{If } A'b' = a'B',\ K_m = Ea'B'/l \qquad [11]$$

$$\text{If } A'b' \neq a'B',\ K_m = \frac{E(A'b' - a'B')}{l \log_e (A'b'/a'B')} \qquad [12]$$

Stability. Stability may be defined as freedom from oscillation (or hunting) caused by a fault in the calculation and not present in the objects being analyzed.

One way to detect instability is to plot the values of D_m, V_m, and F_m as in Fig. 4, and examine the curves for sharp peaks or sharp wiggles. (It already has been pointed out that the sharp peak at P in Fig. 4(*b*) indicates instability.) Another way is to examine the tabulated values of Z_m. If any of these shows a tendency to fluctuate from interval to interval, instability is present. In Table 2 the values of Z_2 fluctuate in this way in intervals 12, 13,

... but the fluctuation does not continue; therefore it ... ignored unless great accuracy is desired. The occurrence ... means that a numerical error has been made, or that ... large. Instability may occur suddenly in the middle of a ... especially if comparatively large external forces or ... are inserted suddenly, or are required to change sign ... as in the case of R_2 of Fig. 3.

... Lengths l and Time Interval Δt. In some problems the unit ... are decided more or less automatically, as in a short rail... train where each car and coupler normally would be chosen ... unit length. If the object is continuous it may be divided ... any number of units (preferably of equal length) depending ... the degree of accuracy desired and the computing personnel ... equipment available.

... the unit lengths have been decided, the time interval ... is chosen to correspond. An unnecessarily small time in... involves a great deal of extra work with little or no increase ... accuracy. Too large a time interval produces instability and ... inaccuracy.

... spring in a diagram such as Fig. 1 or Fig. 3 has a "critical" ... interval which is the time that it would require for a sound ... wave to traverse this particular spring and its associated ... Remembering that sound waves travel in both directions ... at a speed equal to $\sqrt{E/\rho}$ where ρ equals the mass per unit ... the following formulas may be derived for the critical ... interval for spring K_m, which will be called T_m:

... wave motion to right

$$T_m = \sqrt{\frac{W_{m+1}}{12gK_m}} = \frac{1}{19.648}\sqrt{\frac{W_{m+1}}{K_m}} \qquad [13]$$

... wave motion to left

$$T_m = \sqrt{\frac{W_m}{12gK_m}} = \frac{1}{19.648}\sqrt{\frac{W_m}{K_m}} \qquad [14]$$

... lower value of T_m from either Formula [13] or [14] is the ... one.

... Formulas [13] and [14] are applied to Fig. 3 the results are ... follows:

... wave motion to the right, Formula [13]

$$T_1 = 1/1316 \qquad T_2 = 1/1030 \qquad T_3 = \infty$$

... wave motion to the left, Formula [14]

$$T_1 = 1/589 \qquad T_2 = 1/1455 \qquad T_3 = 1/1120$$

The value of ϕ must never be less than unity; otherwise the re... of the numerical calculation will be meaningless because the ... numerical calculation will not progress as fast as the actual sound ... wave. If the value of ϕ is close to unity, instability is ... to occur. The value of ϕ used for Tables 1 and 2 was ... $\div$ 1/3000 = 2.06.

... of Time Interval Δt. The time interval may be changed ... at any line of the numerical calculation, such as ... 2. Ordinarily, this involves only the writing in of new ... for the constants $12\Delta t$ and $\Delta tg/W_m$ used in Formulas [1] ... [5], but this change must be made simultaneously in all ... D_m and V_m of a calculation such as Table 2.

Types of Impact: Practical problems may involve at least ... distinct types of impact as listed and discussed in the ... following. In this discussion the assumption is made that all ... springs are perfectly elastic. Methods for taking account of ... damping and inelasticity will be discussed later on.

The spring that receives the first impact, such as K_1 of Figs. 1 and 3, and K_3 of Fig. 10, will be called the "impact spring."

In some impact problems there is an actual impact spring or cushion on which the ram may strike. For instance, a locomotive bumping into a line of freight cars first encounters coupler springs, or "draft gears," and must compress them in order to move the first car. Such problems will be called "cushioned impact." On the other hand, if a ram strikes a steel rod or anvil directly, there is no true impact spring present. Such problems will be called "direct impact."

A single moving weight, such as a locomotive without cars, or the moving part of a forging hammer, will be called a "ram." A group of objects, such as a railroad train in which each comparatively rigid car is separated from the next by a spring coupler, will be called "a group of separate weights and springs," and must be distinguished clearly from a "long object" such as a long steel rod or a driven pile.

An Important Distinction. In this method a long object is represented conventionally by the same type of diagram that is used for a corresponding group or separate weights and springs. If the impact is well cushioned by a comparatively "soft" impact spring, a long object and a corresponding group of separate weights and springs will act almost exactly alike. On the other hand, if the impact is direct, or if the impact spring is comparatively stiff, surges or minor oscillations may occur in a group of separate weights and springs that would not be present in a corresponding long object. This difference calls for the observance of certain rules as listed in the following section.

Types of Impact

Type I Impact. Cushioned impact between a ram and a group of separate weights and springs: This type of impact presents a diagram similar to Figs. 1 and 3. A locomotive bumping into a number of railroad cars with spring couplers is an example.

Good accuracy is obtainable with this type of problem because the diagram used is likely to be an accurate representation of the actual objects involved. The numerical calculation is similar to Table 2. The accuracy may be increased by decreasing the interval. Ordinarily, a value of ϕ of 3 or 4 will give results accurate within a few per cent if the calculation is not carried through more than 100 intervals.

Type II Impact. Cushioned impact between two groups of separate weights and springs: A problem of this type presents a diagram such as Fig. 10.

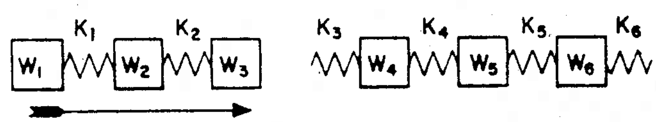


Fig. 10

The numerical calculation is handled as in Table 2 except that positive velocities appear on line O for W_1, W_2, and W_3 instead of for W_1 only. Accuracy obtainable is about the same as for Type I impact.

Type III Impact. Cushioned impact between a ram and a long object: After the long object has been divided as in Figs. 5, 6, 7, and 8, this type of impact presents a diagram similar to Fig. 1 or Fig. 3. The driving of a steel or concrete pile into the ground is an example of this type of impact, because ordinarily a cushion of wood or similar material is used between the ram and the pile. Suitable ground resistance or resistances are introduced and the problem is handled very much like the problem of Fig. 3.

The accuracy obtainable with this type of problem is ordinarily not quite as great as with problems of Type I or Type II unless the impact spring is comparatively soft. This leads to the following practical rule:

Rule. The critical time interval for the impact spring as calculated by Formula [13] or [14] should be at least 1.5 times as large [illegible] interval for any other spring in the system. [illegible] is comparatively stiff it will be found that [illegible] quite small unit lengths [illegible] may [illegible] of computation which in some cases is im[illegible] the rule should be followed as nearly as [illegible] to follow the rule will result in peak forces and [illegible] their true values, but the transfer of [illegible] from the ram to the object struck will still [illegible]

[illegible] impact accuracy may best be increased by re[illegible] lengths. A value of ϕ of about 2 is usually satis[illegible] except the impact spring.

[illegible] Combined impact between two long objects: [illegible] a long object and should be divided into [illegible] same way as the object struck and using ap[illegible] value for [illegible] This will produce [illegible] diagram [illegible] and the remarks given for Type III impact [illegible]

[illegible] Direct impact between a ram and a group of [illegible] A forging hammer which, through [illegible] is allowed to strike its anvil directly is an example of [illegible] Such a hammer might be represented by [illegible] the ram, W_2 the steel anvil, W_3 the concrete [illegible] or special cushion under the anvil, and [illegible] the ground or foundation.

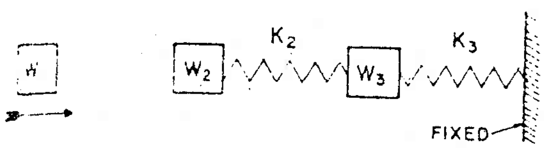


FIG. 11

[illegible] impact spring present the impact will take place [illegible] weights W_1 and W_2. There are two recommended [illegible] the problem:

[illegible] 1. The instantaneous velocities of W_1 and W_2 imme[illegible] after impact occurs may be calculated from Newton's [illegible] considering these two weights only, and these resultant [illegible] may be used to start the numerical calculation such as [illegible] If W_2 is considerably heavier than W_1 (as in a forging [illegible]) this is the simplest and the recommended method be[illegible] a single major impact will occur between W_1 and W_2 [illegible] therefore W_1 may be omitted entirely from the numerical [illegible]. However, if W_2 is lighter than this there may be [illegible] impacts and it is simpler to use Method 2.

[illegible] 2. Arbitrarily insert an imaginary impact spring K_1 [illegible] the numerical calculation can be made like that for Type I [illegible] To assign a suitable spring constant for this imaginary [illegible] spring proceed as follows:

[illegible] $T_m = \phi_m \Delta t$. Making this substitution in Formu[illegible] and transposing, gives

[illegible] motion to right

$$K_m = \frac{W_{m+1}}{12g(\phi_m \Delta t)^2} = \frac{W_{m+1}}{386.04(\phi_m \Delta t)^2} \quad \ldots\ldots [15]$$

[illegible] motion to left

$$K_m = \frac{W_m}{12g(\phi_m \Delta t)^2} = \frac{W_m}{386.04(\phi_m \Delta t)^2} \quad \ldots\ldots [16]$$

[illegible] value of K_m from either Formula [15] or [16] is the [illegible] and a value of ϕ_m of 1.2 is recommended for the [illegible] impact spring mentioned.

[illegible] the numerical calculation the value of C_1 may become [illegible] When this happens it means that W_1 and W_2 have bounced apart; therefore F_1 should be given the value of zero until C_1 again becomes positive.

With this type of problem accuracy may be increased by decreasing the time interval Δt. The smaller Δt is made, the stiffer will be the imaginary impact spring as calculated from Formula [15] or [16], and the higher will be the values obtained for F_1. From this it follows that the values for F_1 and C_1 given by the numerical calculation are not correct or true values.

Sometimes an impact spring may be obtained less arbitrarily by borrowing elasticity from the weights involved as explained in connection with Fig. 9. Even if this is done the values given by the numerical calculation for F_1 and C_1 will have doubtful accuracy.

If true values must be had for the forces at or near the point of impact and immediately following the beginning of impact, they must be obtained by other means. A suitable method is explained in connection with Type VIII impact, Method 2, which involves dividing some or all weights into very small unit lengths.

Type VI Impact. Direct impact between two groups of separate weights and springs: This may be handled like Type V except that the impact occurs at a different place.

Type VII Impact. Direct impact between a ram and a long object: This type of impact may be handled by dividing the long object into unit lengths as shown in Figs. 5, 6, 7, and 8, and then introducing an imaginary impact spring, thus making the numerical calculation become similar to that of Type V impact. Formulas [15] and [16] are used to determine the spring constant for the imaginary impact spring as in Type V impact.

This method has some important shortcomings as follows: The numerical values of C_1, F_1, Z_1, D_2, C_2, F_2, Z_2, and V_2 are likely to be quite incorrect. In the numerical calculation W_2 serves merely as a means of transmitting energy and momentum, and its motion as given by the calculation is not likely to be correct. Numerical values of F_m and V_m starting with F_3 and V_3 will show peak values that may be as much as 25 per cent above the theoretically correct ones, and will tend to oscillate above and below the theoretically correct values. If halving the unit lengths l and making a corresponding change in the interval Δt and in the stiffness of the imaginary impact spring as per Formulas [15] and [16] results in more rapid oscillations, it may be concluded that these minor oscillations do not, in fact, exist, and smooth curves may be plotted eliminating them. Similar minor oscillations show up clearly in the velocity curves of Fig. 4. In the case of Fig. 4 these are true oscillations because W_2 and W_3 are separate and distinct weights. However, if W_2, K_2, and W_3 were intended to represent a single long object, these minor oscillations would have to be investigated further in the manner explained in the foregoing.

In applying this numerical method to Type VII impact it should be borne in mind that from a practical standpoint its tendency to produce false minor oscillations, and thus give peak stresses and velocities higher than theoretical, is not entirely a disadvantage because of the uneven stress distribution throughout a cross section that is almost sure to occur with this type of impact. If values closer to theoretical are desired for forces and velocities, they may be obtained at the cost of considerable extra computation by using Method 2 listed under Type VIII impact which follows.

Type VII impact has been discussed by Donnell (2), and a knowledge of his method is very helpful.

Type VIII Impact. Direct impact between two long objects: Method 1. This type of impact may be handled like Type VI except that the two long objects first must be divided into weights and springs in accordance with Figs. 5, 6, 7, and 8. The remarks as to accuracy of results made in connection with Type VII apply here also except that the subscripts must be changed to

...rrespond. If K_m is the imaginary impact spring then the ...lues that are definitely incorrect are as follows:

D_{m-1}	C_{m-1}	F_{m-1}	Z_{m-1}	V_{m-1}
D_m	C_m	F_m	Z_m	V_m
D_{m+1}	C_{m+1}	F_{m+1}	Z_{m+1}	V_{m+1}

Other values tend to show the same minor oscillations as noted ...der Type VII.

Method 2. If both long objects are divided into unusually small ...it lengths, the imaginary impact spring may be chosen in ...cordance with the following rule without introducing any im...rtant amount of elasticity into the system:

Rule. The value chosen for ϕ_m for the imaginary impact spring ...use in Formulas [15] and [16], should be 1.5 times as great as ...e largest value of ϕ_m applying to any other spring in the system. (This rule is the same as the rule given under Type III impact, ...t is stated in different words.)

If the foregoing rule is followed, the false minor oscillations ...ntioned under Type VII impact will be almost completely ...minated, and the forces and velocities will be very close to the ...oretical ones. The smaller the unit lengths used, the greater ...be the accuracy.

Method 2 furnishes a means of getting very accurate results on Type VII problem also. It is merely necessary to choose such ...all unit lengths that the ram as well as the long object may be ...ided into small sections. This reduces a Type VII problem to Type VIII problem. In order to save computation time this ...cess ordinarily would be used only to check maximum forces ...d velocities immediately after impact, but if the necessary com...tational capacity is available there is no reason why it should ...be used for the complete numerical calculation with resulting ...h accuracy.

Method 2 also may be used to determine accurately the ...esses at the point of impact in Type V and Type VI problems. Type VIII impact has been discussed by Donnell (2), and a ...wledge of his method is very helpful.

IRREGULARITIES

...arious irregularities may be introduced into the calculation, ...h as those illustrated in Table 2 where the resistances R_7 and ...changed sign whenever the corresponding weight changed its ...ection of motion, and where F_1 became zero and remained zero ...ead of following Formula [3] into the negative range. Sudden ...gularities such as the foregoing are called "boundary condi...s." It is possible to devise an almost endless number of irregu...ties that can be handled successfully by this numerical method. ...ever, it will pay to introduce only such irregularities as will ...erially improve the accuracy of the calculated results, be...e the extra work involved may be considerable. Examples of ...us types of irregularities follow.

...riable Resistances, R_m. In some problems the external re...ances R_m may vary from interval to interval according to some ...ite formula. For instance, R_m might vary linearly with the ...ity v_m. In this case R_m would be computed for each time ...val and listed in a separate column, using the formula

$$R_m = \psi v_m \qquad [17]$$

...e ψ = a suitable constant.

...rmula [17] is merely illustrative and must be modified to suit ...roblem in hand. For instance, R_m might vary as the square ...r as the displacement D_m, etc.

...mping. Various types of damping can be handled success... The simplest type is the introduction of frictional re...ces as in Table 2. Formula [17] is also a form of damping. ...steresis loop suitable for a spring that acts both in compression and in tension can be produced by using the following formula instead of Formula [3]

$$F_m = K_m\left[C_m + \theta\left(\frac{C_m - c_m}{\Delta t}\right)\right] \qquad [18a]$$

where θ = a suitable constant.

A hysteresis loop suitable for a spring that acts only in compression can be produced by using the following formula instead of Formula [3]

$$F_m = C_m K_m\left[1 + \theta'\left(\frac{C_m - c_m}{\Delta t}\right)\right] \qquad [18b]$$

where θ' = a suitable constant.

Coefficient of Restitution. Occasionally materials, such as rubber, which have a low coefficient of restitution, are used as springs or "cushions." Such materials when tested statically in the laboratory may give force-deflection curves of the type shown in Fig. 12.

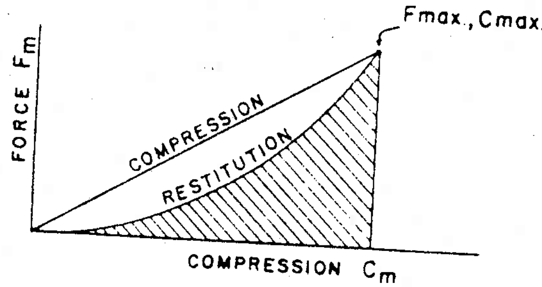


FIG. 12

The compression part of the curve may be straight enough so that the use of Formula [3] is justified up to the maximum point F_{max}, C_{max}, Fig. 12. For restitution, an equation is needed whose graph passes through the origin and the point F_{max}, C_{max}, and whose shape is similar to the restitution part of the curve shown in Fig. 12. Equation [19] satifies these conditions reasonably well

$$F_{mr} = \frac{F_{max}}{(C_{max})^u}(C_m)^u \qquad [19]$$

in which

$u = (2/\eta^2) - 1$

F_{mr} = force exerted by spring m in time interval n during restitution (or recoil)

F_{max} = maximum value of F_m

C_{max} = maximum value of C_m

η = coefficient of restitution for spring m

During restitution only part of the energy of compression is returned. This energy return varies as the square of the coefficient of restitution η, and is represented by the shaded area in Fig. 12.

In making the numerical calculation the force F_m is calculated by means of Formula [3] until the value of C_m begins to decrease. At this point in the calculation the maximum values attained by F_m and C_m, namely, F_{max} and C_{max}, Fig. 12, are substituted in Formula [19] and this formula is then used to compute F_{mr} until C_m returns to zero.

A somewhat awkward problem arises as to what is to be done if C_m again begins to increase before returning to zero. If only a slight hesitation is involved it is better to stick to Formula [19] until C_m does return to zero, rather than to make additional changes of formula.

Plastic Flow. The force F_m calculated by Formula [3] may become so great that plastic flow occurs, as in a forging process or

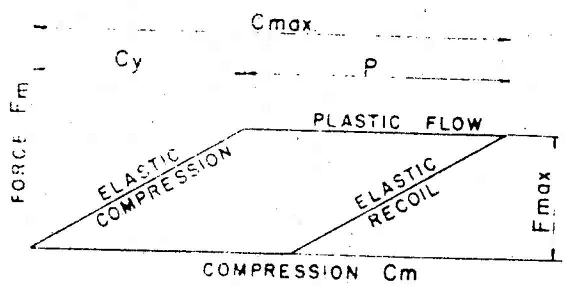


Fig. 13

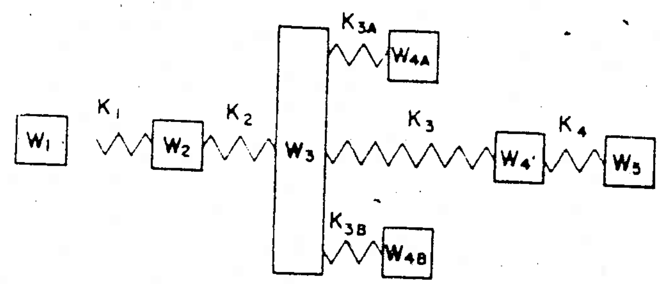


Fig. 14

when a pile is driven into the ground. This condition corresponds to a deflection curve of the type shown in Fig. 13.

During elastic compression, Formula [3] would be used. During plastic flow F_m may remain practically constant. As soon as the spring begins to expand, as indicated by a decrease in the value of C_m, Formula [20] would be used

$$F_m = K_m(C_m - P) \quad \ldots\ldots \quad [20]$$

where

P = amount of plastic flow, in.
$= C_{max} - C_y$
C_y = value of C_m at yield point, in.

In the numerical calculation it may be convenient to list the computed values of $C_m - P$ in a separate column.

Graphical or Tabular Relationships. Sometimes it is not possible or convenient to express the relationship between F_m and C_m as a formula such as Equation [3], [18a], [18b], [19], or [20], but graphs or tabular data may be available showing the relationship. If the numerical calculation is performed by hand, the values for F_m corresponding to each value of C_m can be read directly from the graphs or obtained from the tabular data by interpolation. This process may involve the complication that the curve of restitution may have to be adjusted so that it will start at the maximum point reached by the curve of compression, as was done with a curve in Formula [19]. Some electronic digital calculators can perform a similar operation based on tabular data fed into the machine ahead of time.

Gravity. In problems involving vertical motion, the static forces due to gravity ordinarily may be neglected. If the weights are large, as in a forging hammer, the gravity forces may be inserted as negative resistances R_m, with consequent initial compressions C_m and initial forces F_m which must appear on line O of the numerical calculation. Alternately, the static effects of gravity may be added algebraically after the numerical calculation of the dynamic forces has been completed. In the latter case the following rule should be observed in making the dynamic calculation:

Rule. If any particular spring K_m is not designed to take tension, nevertheless if C_m becomes negative F_m also must be allowed to become negative, but its negative value must not be allowed to exceed the positive static force F_m caused by gravity alone.

When, as a final step, the static-gravity forces are added algebraically to the dynamic forces, the negative and positive values of F_m may cancel each other having a net value of F_m equal to zero.

Branched Systems. A simple branched system is shown in Fig. 14. Such systems may be handled readily by this numerical method. Values for D_m, C_m, F_m, Z_m, and V_m may be calculated for all weights and springs of Fig. 14 by Formulas [1] to [5], with the exception of values for Z_3. There are four springs acting on W_3; therefore $Z_3 = F_2 - F_3 - F_{3A} - F_{3B}$. Similar adaptations of the method may be made for more complicated systems.

Checking the Numerical Calculation

Plotting computed values, as in Fig. 4, is an excellent way of checking accuracy. An error will show up as a sudden irregularity in the curves. The speed with which the actual stress or sound wave progresses also will be observable, and in some problems this may be used as a check.

Repeating the numerical calculation with a smaller time interval or a smaller unit length is another method of checking.

The simplest numerical check is based on the "law of conservation of momentum," which states that the total momentum as given by the following expression always must remain constant

$$\text{Total momentum} = \Sigma W_m V_m/g + \Sigma R_m t \quad \ldots\ldots [21]$$

In using this check, R_m must be interpreted to mean all external forces such as R_2, R_3, and F_3 of Fig. 3. The total momentum may be computed for any line of the numerical calculation such as Table 2.

A more complete numerical check is based on the "law of conservation of energy," which states that the total energy as given by the following expression must always remain constant

$$\text{Total energy} = \sum \frac{W_m}{2g}(V_m)^2 + \sum \left(\frac{C_m}{12} \times \frac{F_m}{2}\right)$$
$$+ \sum \text{energy lost externally or as heat} \quad \ldots [22]$$

The necessary values for substitution in this formula will be found in a tabulation such as Table 2, but some of them may not be at once obvious. This energy check is not as accurate as the momentum check, especially for the first four or five intervals. Usually a check of this kind is made only at the end of the calculation.

Acknowledgment

Thanks are due to Mr. William Heising, Assistant Manager, Electronic Data Processing Service, International Business Machines Corporation, New York, N. Y., and to Dr. W. E. Milne, Head of the Department of Mathematics, Oregon State College, Corvallis, Ore., for reading the manuscript and for helpful comments; also to my son and daughter for frank criticism and material help.

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The foregoing references include extensive bibliographies.

Appendix

The following notes will be of interest to those who wish to investigate fundamentals:

1 Formulas [1] to [5] are extrapolation formulas. Other formulas, as well as iterative procedures, may be devised, but Formulas [1] to [5] are simple, surprisingly accurate, and have an inherent and important tendency to produce stability even when comparatively large time intervals are used.

2 A single weight and spring will obey the laws of harmonic motion. Any values may be assumed, but the simplest case for purposes of calculation is arrived at as follows:

Omitting subscripts as unnecessary, let D, C, and K be expressed in the same units as F, Z, and V, thus eliminating the constant 12 from Formulas [1], [13], [14], [15], [16], and [22]. Let $W = g$ and let $K = 1$. Then Formulas [1] and [5] become

$$D = d + v\Delta t \qquad [1a]$$

$$V = v - D\Delta t \qquad [5a]$$

From these formulas we readily may calculate values for sin t and cos t by letting $V = 1.000$ at $n = 0$, and considering that t is measured in radians. Formula [14] then gives $T = 1$. If $\phi = 5$, then $\Delta t = 0.2$, and it will be found that the calculated values of D approximate the tabular values of sin t. However, the calculated values of V will not closely approximate the tabular values of cos t after the first interval unless the angles used in looking up the tabular values of cos t are taken as $1\frac{1}{2}\Delta t$, $2\frac{1}{2}\Delta t$, $3\frac{1}{2}\Delta t$, etc. This indicates that the values of V tend to be "out of phase" by about half an interval.

3 A "classical" impact problem is that of an infinite weight or ram striking directly on the end of a uniform rod of infinite length. If this problem is represented as in Fig. 1, with all springs and all weights exactly equal, excepting the ram W_1, then a value of $\phi = 1$ will give results that agree exactly with theory. Forces F_m and velocities V_m for each weight and spring will jump instantaneously to their maximum and theoretical values, and this action will progress down the length of the rod with exactly the speed of sound. This result will be obtained no matter what value of l is used. If, however, a larger value is used for ϕ, the values of F_m and V_m will oscillate above and below the values obtained with $\phi = 1$, but, nevertheless, the peak values of F_m and V_m will travel down the rod with a speed closely approximating the speed of sound. If ϕ is given a value of about 3 or more, the calculated results will correspond closely to the action of a group of separate, and equal, weights and springs.

Discussion

W. P. Heising.[4] The equation of motion for free longitudinal vibrations in a continuous bar is

$$mD_{tt} - (KD_x)_x = 0$$

[4] Assistant Manager, Electronic Data Processing Service, International Business Machines Corporation, New York, N. Y.

where

D = longitudinal displacement of a thin disk from its reference point
m = mass per unit length
K = "spring constant" or Young's modulus times cross-sectional area
x, t are independent variables for position and time, respectively, with subscripts denoting partial differentiation

This partial differential equation is approximated by the difference equation

$$\begin{aligned} m(\Delta t)^{-2}[D(x, t + \Delta t) - 2D(x, t) + D(x, t - \Delta t)] \\ = (\Delta x)^{-2}\{K(x + 0.5\Delta x)[D(x + \Delta x, t) - D(x, t)] \\ - K(x - 0.5\Delta x)[D(x, t)D(x, t - \Delta t)]\} \end{aligned}$$

If the D are known for two successive time steps (equivalent to known initial positions and velocities) the foregoing differential equation can be integrated one time step at a time into the future. It is instructive to consider the simple case of a uniform bar with $c = (K/m)^{0.5}$. The solution of the partial differential equation is

$$D = f_1(x - ct) + f_2(x + ct)$$

In this example with fixed ends the solution of the difference equations with N space intervals yields the correct characteristic modes of sine waves traveling at a velocity V. We find, however, that instead of the velocity being independent of the mode (frequency), that

$$v = c\,\frac{\text{arc sin }[\phi^{-1}\sin(n\pi/2N)]}{\phi^{-1}(n\pi/2N)}$$

where

$$\phi = \Delta x/(c\Delta t);\ n = 1, 2, \ldots\ldots N - 1$$

We have thus the surprising result that if $\phi = 1$ (i.e., $c = \Delta x/\Delta t$), the solution of the difference equation is an absolutely exact solution of the partial differential equation with $v = c$ for all modes. If $\Delta t > \Delta x/c$, v becomes imaginary for some large n, indicating exponential behavior in time of the high-frequency modes. These cause the amplitude to increase without limit; thus the numerical method is unstable.

$\Delta t < \Delta x/c$ gives a less accurate solution to the case of the uniform bar, the highest frequencies traveling near $2/\pi$ or 64 per cent of the true sonic velocity. This leads to dispersion. If $\Delta t \to 0$, the solution approaches the behavior of N discrete identical springs. In all the stable solutions energy and momentum are conserved.

In the actual physical problems, the author considers nonuniform bars and friction. The significance of ϕ is qualitatively explainable in terms of the behavior found for the uniform bar. Where there is a ram striking a continuous bar, an effective spring is inserted between them. In order that the solution of the difference equations behave properly, we wish $\phi_{\text{spring}} \gg 1$ and ϕ_{bar} approximately equal to, but not less than unity. Where the impact is cushioned or "soft," these conditions can be made to hold readily with quite practical values for Δx and Δt.

Some consideration has been given to the use of alternative numerical integration formulas. The application of the author's formulas to the case of a body in simple harmonic motion gives an apparent frequency higher than the true frequency by a factor of

$$\left[1 + \frac{1}{24}\left(\frac{2\pi\Delta t}{T}\right)^2\right]$$

Investigation shows that if Δt is chosen so that less than 30 points per cycle are used, better or comparable accuracy is obtained by his formulas for the same effort, when one considers that each integration step is twice as laborious in more elaborate methods. If more elaborate difference approximations are adopted for D_{tt}, it would appear desirable to do the same for $(KD_x)_x$.

In most of the problems considered in the paper, the necessary physical assumptions for friction and cushioned impact are sufficiently crude, that refinements in the difference approximations do not seem justified.

W. E. Milne.[5] The problem considered in the paper deals with the transmission of waves along a linear body, either continuous or made up of discrete parts elastically connected. The motion is subjected to a force of resistance. The continuous case leads to a partial differential equation of the type

$$\rho \frac{\partial^2 D}{\partial t^2} = \frac{\partial}{\partial x}\left(k \frac{\partial D}{\partial x}\right) \pm R$$

where D is displacement at time t at distance x from one end, ρ is mass per unit length, k depends on the spring constant, and R is the resisting force. In the special case where $R = 0$ and ρ and k are constant this reduces to the familiar wave equation, which is readily solved. However, the author deals with variable ρ and k and nonzero resistance, a case for which an analytic solution is not obtainable.

He resorts therefore to an approximate solution by considering the body as a train of discrete parts, for each of which he sets up the equation of motion, in this case an ordinary second-order differential equation in terms of time. This is a standard procedure in the numerical treatment of partial differential equations.

The following comments deal entirely with various ways of carrying out the numerical integration of this system of ordinary equations. Since time did not permit working out the five different methods of the illustrative example which the author solved in his paper, the writer has selected a simpler problem which is easier to handle, and also for which the correct solution can be found so that the approximate results can be checked. The problem is this: A unit mass is drawn toward the origin by a force equal to four times the distance from the origin and subject to a resisting force of unit magnitude. The equations of motion for this problem are

$$\frac{dD}{dt} = V$$

$$Z = \frac{dV}{dt} = -4D \pm 1$$

(+ if V is negative; — if V is positive). We take $\Delta t = 0.1$ and start with $D = 0$, $V = 1$ at $t = 0$.

Method 1. This is the method used by the author and is based on the equations (cf. Equations [1] and [5] of the paper)

$$D = d + v\Delta t$$

$$V = v + Z\Delta t$$

The solution using these equations is shown under Method 1 on the accompanying computation sheet. Comparison with the true values shows fair agreement for D, but poor agreement for V.

Method 2. This is similar in spirit to Method 1, but uses the equations

$$D = d + V\Delta t$$

$$V = v + z\,\Delta t$$

Here the error in D is about the same as for Method 1, but V is somewhat better, mainly because the effect of the sign change in the resistance is taken into account one step earlier.

Method 3. This is not a practical computational method but is given for illustration. The values are obtained by using both Method 1 and Method 2, and then taking the average.

Method 4. This is the practical way of carrying out Method 3

[5] Head, Department of Mathematics, Oregon State College, Corvallis, Oregon.

COMPUTATION SHEET

Method 1

	D	V	Z
0	0	1.0000	−1.000
0.1	0.10000	0.8600	−1.400
0.2	0.18600	0.6856	−1.744
0.3	0.25456	0.4838	−2.01[illegible]
0.4	0.30294	0.2026	−2.212
0.5	0.32920	0.0309	−2.317
0.6	0.33229	−0.2020	−2.329
0.7	0.31209	−0.2268	−0.248
0.8	0.28941	−0.2426	−0.158
0.9	0.26515	−0.2487	−0.061
1.0	0.24027	−0.2448	−0.039

Method 2

t	D	V	Z
0	0	1.0000	−1.000
0.1	0.09000	0.9000	−1.360
0.2	0.16640	0.7640	−1.666
0.3	0.22614	0.5974	−1.905
0.4	0.26683	0.4069	−2.067
0.5	0.28685	0.2002	−2.147
0.6	0.28540	−0.0145	−0.142
0.7	0.28253	−0.0287	−0.130
0.8	0.27836	−0.0417	−0.113
0.9	0.27306	−0.0530	−0.092
1.0	0.26684	−0.0622	−0.077

Method 3

0	0	1.0000	−1.000
0.1	0.09500	0.8800	−1.380
0.2	0.17620	0.7248	−1.705
0.3	0.24035	0.5406	−1.962
0.4	0.28489	0.3348	−2.140
0.5	0.30803	0.1156	−2.232
0.6	0.30885	−0.1083	−1.236
0.7	0.29731	−0.1277	−0.189
0.8	0.28388	−0.1422	−0.136
0.9	0.26911	−0.1508	−0.077
1.0	0.25356	−0.1535	−0.058

Method 4

0	0	1.0000	−1.000
0.1	0.09500	0.8800	−1.380
0.2	0.17620	....	−1.705
0.3	0.24035	....	−1.961
0.4	0.28489	....	−2.140
0.5	0.30803	....	−2.232
0.6	0.30885	....	−1.235
0.7	0.29732	....	−0.189
0.8	0.28300	....	−0.136
0.9	0.26912	....	−0.076
1.0	0.25358	....	

Method 5

0	0	1.0000	−1.000
0.1	0.09406	0.8812	−1.376
0.2	0.17450	0.7276	−1.698
0.3	0.23814	0.5451	−1.953
0.4	0.28245	0.3410	−2.130
0.5	0.30566	0.1234	−2.223
0.6	0.30889	−0.0143	−0.236
0.7	0.30630	−0.0374	−0.225
0.8	0.30162	−0.0562	−0.206
0.9	0.29504	−0.0755	−0.180
1.0	0.28667	−0.0919	−0.148

True Solution

0	0	1.0000	−1.000
0.1	0.09435	0.8807	−1.377
0.2	0.17497	0.7264	−1.700
0.3	0.23866	0.5430	−1.955
0.4	0.28286	0.3380	−2.131
0.5	0.30581	0.1196	−2.223
0.6	0.30877	−0.0110	−0.235
0.7	0.30651	−0.0341	−0.226
0.8	0.30200	−0.0559	−0.208
0.9	0.29541	−0.0754	−0.182
1.0	0.28701	−0.0919	−0.148

it is actually easier than either Method 1 or Method 2, and generally gives better results than either. The first step is computed by Method 1 and by Method 2 and the average is taken. From then on the computation proceeds by the formula

$$D = 2d - d^* + z(\Delta t)^2$$

where d^* is the value preceding d. Thus to compute D for $t = 0.5$ we have

$$D = 2(0.28489) - (0.24035) + (-2.140)(0.01)$$

In computing Z for the line where the sign of the resistance changes, the value of R was replaced by zero, the average of $+1$ and -1.

Method 5. This is the one explained for first-order equations in a previous work by the writer.[6] Because each line is rechecked it takes more work than the other methods but it is much more accurate.

Comparison with the true solution shows that Method 5 gives decidedly better results than any of the other methods, and this is especially true for the values of V.

Effect of sign change in R. At a point where . changes sign, R also changes sign and produces a discontinuity in Z. The approximate formulas used in the foregoing methods are liable to be highly inaccurate when discontinuities occur, and this error is propagated, sometimes with cumulative effect, far past the point where this sign changes. Instead of using formulas for this step it is better to perform the integrations graphically. By plotting V against t we can tell approximately where in the next interval the value of V will be zero. At this point the magnitude of Z changes by the amount $2R$. Even a very rough graph will give a better estimate for the next V and D than use of the formulas. This was done in Method 5.

The easiest method is probably Method 4, and it is usually also somewhat more accurate than 1 or 2. The most accurate method of those mentioned here is Method 5. This method has two other virtues: (a) It contains in itself a check by which one can estimate how big the error probably is. The other methods do not tell how far off the result may be.[7] (b) In using Method 5 we do not need to worry about the stability of our numerical solution. For the other methods an improper choice of the interval Δt may cause the process to diverge. In Method 5 it is easy to see whether the process converges. Finally, Method 5 gives much more accurate values of V, and, although we are not interested in V itself, the place where V changes sign is very important, as we see by comparing Methods 1 and 2. Hence we need to have reasonably accurate values of V.

[6] Author's reference (7), articles 8, 9, 10, and 11, pp. 24–30.

[7] Author's reference (7), articles 10 and 11.

Author's Closure

The discussions by Dr. Milne and Mr. Heising are important contributions to our knowledge of this subject. Mr. Heising gives mathematical background that is not included at all in my paper, and which should prove valuable to anyone who wishes to investigate the problem thoroughly. Dr. Milne proposes certain alternative methods for the purpose of increasing accuracy.

Dr. Milne's Methods 2, 3, and 4 are basically identical with my method (which he calls Method 1) except in the handling of the frictional force. The formula he uses for D in Method 4 is a combination of the two formulas he uses for my method. This combination can easily be made if it is remembered that in Dr. Milne's notation $v = v^* + z\Delta t$, and $v^*\Delta t = d - d^*$, where d, v, and z indicate values in time interval $n - 1$, and d^* and v^* indicate values in time interval $n - 2$. Dr. Milne's problem involves rapid frictional damping applied directly to the ram. Accurate results on this problem can be obtained by my method by using a smaller time interval.

Dr. Milne's Method 5 uses a correction formula to check or correct each step in the calculation, and is designed to give accuracy specifically on problems involving the idealized condition of springs entirely without weight and weights entirely without elasticity. My method is suitable for such problems if comparatively high values of ϕ are used, and for problems involving partially or completely distributed weight and elasticity if lower values of ϕ are used.

In closing it may be well to point out that torsional problems can be handled in a very similar manner.

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