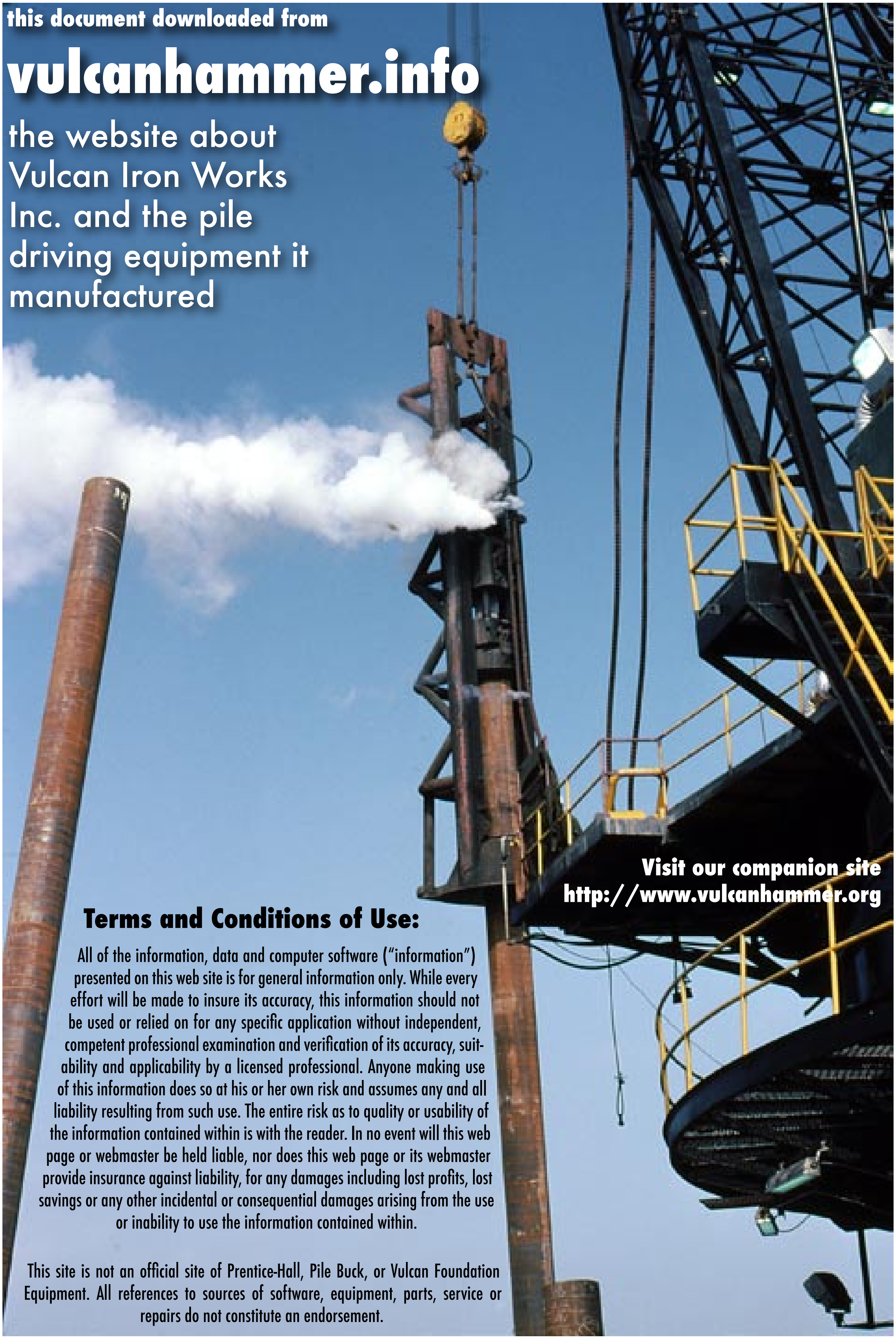

APPLICATIONS FOR NEW RESEARCH
FOR PILE SUPPORTED MACHINE FOUNDATIONS

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SYMPOSIS

The use of piling for machine foundations can add flexibility for the designer, help solve special problems, and possibly reduce costs. A very complete method of analysis is presented with great flexibility in options available as well as a catalogue of very accurate pile models. A design for a power plant using the method is related as an example.

Keywords: Pile Foundation; Machine Foundation; Pile Analysis; Laterally Loaded Piling; Dynamic Loads, Vibrations.

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INTRODUCTION

Foundation Characteristics

Many machine foundations are a concrete block of massive proportions and may be assumed rigid for analysis and design. It follows that the foundation may be accurately modeled as a discrete mass with six degrees of freedom, three translational and three rotational, with respect to orthogonal coordinates, usually aligned parallel to some convenient axis of the foundation. The parameters of the dynamic system are thus inertial or mass, damping, stiffness and forcing function as determined by the type of machine and the nature of its operation.

Although mechanical adjustments, operating procedures, period of operation, or some other machine parameter may be altered to improve characteristics of the foundation-machine assemblage for better performance, the properties of the operating machine are usually accepted as specified parameters of the design.

The parameters of energy dissipation mechanisms in the foundation, usually modeled as viscous damping, may be estimated or evaluated for the system for analysis. For some types of analyses an estimate of damping is required, for others damping may be neglected. In addition to the usual mechanisms of damping in structures, the dominant contribution is from soil-structure interaction. Although energy dissipators may be installed, damping is not usually considered as a design variable but rather as a factor in the analysis resulting from other design variables.

Mass and geometry determine the inertial terms in the dynamics equations. An initial layout of the machine, with consideration of a dynamic load factor or addition to the static force for estimation of loads under operating conditions, allows calculation of bearing pressure for preliminary computations. These pressures are usually assumed uncoupled and separately estimated for vertical and rocking components, sliding and torsional motions would be considered at a later stage in the design. Thus, allowable soil bearing pressure would determine the geometry, and to some extent, the mass. Refinement of the computations may require adjustment in geometry but the layout would, in the main, be considered fixed. The foundation may be varied in thickness and pockets may be left within the block to adjust for mass and mass moment of inertia. This may be considered the prime design variable, that is, the most well understood and easily varied parameter in the system. This is especially true since the foundation may later be "tuned" or its frequency adjusted by adding mass by pouring concrete into the pockets.

If the foundation has no piling, the ratio of load to deflection in each of the six degrees of freedom may be computed from soil and bearing area data. There are a variety of suggestions or theories for this computation in addition to experimental or field measurements. These "spring constants" or uncoupled stiffness coefficients, are usually the best approximation to a linear stiffness. The models used in the computation vary from simplistic to extremely sophisticated, with the best for use in design, given due consideration for the variability and usual lack of sufficient data or low confidence level in information, being somewhere in between. As research and field data increase and improve, however, more sophisticated models will become more practical. The foundation stiffness, similar to damping, is evaluated for analysis but not usually considered as a design variable.

The machine foundation designer then may adjust damping or stiffness with limits, but mainly works with geometry and mass, which are coupled, within the constraints of the soil base on site or as altered or filled. Computations are further complicated by mass coupling or virtual mass. This is some portion of the soil which acts in concert with the concrete block as an apparent added mass. Estimation of this variable places mass in the same category as damping and stiffness as values requiring a "best estimate" for design. These methods have been well outlined by Richart[1].

Pile Foundations

Introduction of piling in the foundation system will affect damping, mass and stiffness[2]. In addition, stiffness becomes a quantifiable design variable. The net result is that the designer has better control over at least one more variable, stiffness, perhaps a more accurate but certainly different model for computation of added mass, and may often use a less massive foundation. Reinforced concrete piling are quite competent for axial loads and in flexure to take lateral components and are most resistant to the many deleterious elements in the air-soil-water interactive zone. In addition, concrete piles can be cast to any cross-section and length desired; precast or poured in place; prismatic, stepped or tapered; reinforced or prestressed; spliced or cut; and designed as drilled caissons or utilizing standard production model driven piles. The reinforced concrete cap forms a rigid joint with the pile and since the bearing capacity of the single pile determines load capacity, the size of the foundation or cap is controlled by type of pile, pile diameter, pile spacing, and number and pattern of piles. The design of the cap itself varies only slightly, the reactive forces being concentrated at the pile locations rather than distributed. The added mass of soil may be approximated as some percentage of that enclosed by the pile pattern above the first inflection point of the piles, probably between 30% and less than 50%. Estimation of damping is no more concise than the previous case and is primarily due to energy dissipated in pile-soil interaction. Although the foundation may be cast in contact with soil, unless the designer has reason to differ, it is assumed that resistance to lateral displacement, sliding or torsion, will be provided solely by piles. Thus frictional contact is deemed negligible. In addition, the pile cap may be cast on poor bearing soils or elevated as a platform. Thus the method of analysis presented has a wide range of applicability, from competent terrain to offshore sites, and for any type of machinery.

THEORY

The undamped free vibrations are expressed by the equations of motion in 6 kinematic degrees of freedom

$$[M]\{\ddot{\Delta}\} + [S]\{\Delta\} = \{0\} \tag{1}$$

where the 6 Δ_i are linear displacements corresponding to the coordinates shown in Fig. 1, 3 translational and 3 rotational small amplitudes, $\ddot{\Delta}_i$ indicates differentiation with respect to time and are the acceleration components, the components of the diagonal mass matrix are mass and mass moment of inertia corresponding to translational and rotational degrees of freedom, and [S] is the stiffness matrix. The concern here is primarily with the stiffness matrix. Once Eq. 1 is formed the six frequencies ω_i and mode shapes $\{\phi\}_i$ may be determined with the quite useful and valid assumption that damping may be taken in a form where it is a function of

[M] and/or [S], see discussion in Reference 3.

If damping is desired to be included in computations, a percentage of critical damping in a viscous model in each mode may then be assumed, assessed, or measured and in this form is much more useful and understandable. In the absence of better data, damping would probably be assumed equal in each mode and modestly estimated at about 10% for fully embedded piles, less for platforms. If the circular frequency in mode i is designated ω_i and the percentage of critical as a decimal ξ_i the damping coefficient in mode i is $2\,\xi_i\omega_i$; this may later be converted to the damping force in mode i or viscous damping coefficients to be added as a damping force to Eq. 1, see Reference 4 for a computational method for damping as well as a method of modal numerical integration.

Stiffness

The stiffness matrix is a property of the structure. A coefficient S_{ij} may be defined as the force at i due to a unit displacement at j with all other displacements being zero. The subscripts i and j refer to the kinematic degrees of freedom, which coincide with the coordinate system in Fig. 1. Thus, S_{15} is a force in the direction of axis U_1 due to a unit rotation $\Delta_5 = 1$, which is about the U_2 axis. Further, the coefficients S_{i5} in column 5 of the stiffness matrix are the complete set of forces caused by the $\Delta_5 = 1$ with all other $\Delta_i = 0$. The units of the S_{ij} are mixed, force per unit translation (kN/mm or lb/in.), force per unit rotation (kN/rad), moment per unit translation (kN-mm/mm) or moment per unit rotation (kN-mm/rad). The inverse of the stiffness matrix is a flexibility matrix $[D] = [S]^{-1}$ where its coefficients d_{ij} are the displacements at i due to a unit force at j with all other forces being zero. These coefficients may be determined more easily by testing; thus if a force were applied to a pile foundation in one of the coordinate directions, say along Δ_1, the 6 components of displacement may be measured and divided by the magnitude of the force applied to form the first column or vector $\{d\}_1$ of the flexibility matrix. If any force or load component were graphed versus a convenient component of displacement for changing increments of load during the test, the resulting curve would probably not be linear or elastic, thus the slope d_{ij} would not appear to be a constant quantity. Since the analysis is most easily and reasonably performed with the assumption of a linear elastic system, that is the cofficients S_{ij} or d_{ij} being constants, it is necessary to make assumptions to simplify the system. For the coefficients obtained by test, for example, it would be possible to cycle the load in the neighborhood of magnitude eventually expected and to use the resulting graph to obtain a secant modulus.

The stiffness of the foundation is a function of all components; soil properties, rigidity of the cap, and properties of each pile, and their configuration as a group. The foundation stiffness is, in fact, the sum of the stiffnesses of all its components and since the cap is assumed to be rigid, it is the sum of the stiffness contributions of each pile interacting with the soil. The stiffness of each pile then, in directions parallel to the kinematic degrees of freedom of the foundation, is needed. Thus

$$[S] = \sum_{k=1}^{n} [S']_k = [S']_1 + [S']_2 + \ldots + [S']_n \qquad (2)$$

where $[S']_k$ is the stiffness of pile k of n-piles in the foundation coordinates[2].

Pile Stiffness

The pile stiffness is first determined in coordinates local to the pile. Extensive use of Reference 2 is useful to this section. With origin at the centroid of area at the pile head, these are formed by the longitudinal axis and the principal axes in bending. Subsequently, the pile stiffness is transformed twice, once in rotation parallel to the axis of the foundation stiffness, and secondly to account for the pile position in the foundation with respect to a coordinate center of the foundation axes. The pile stiffness may be expressed in the relationship

$$\{F\}_k = [b]_k \{x\}_k \tag{3}$$

where the b_{ij} relate forces F_i to displacements x_j. The coordinate system is a triad similar to that used in Fig. 1. The form of $[b]_k$ is a sparse matrix, the diagonal elements and $b_{24} = b_{42}$ and $b_{15} = b_{51}$ being the only nonzero although these too may be zero depending on boundary or end conditions. Determination of the coefficients of [b] follows later.

The transformations necessary are

$$[S']_k = [c]_k [a]_k [b]_k [a]_k^T [c]_k^T \tag{4}$$

where the rotation transformation matrix $[a]_k$ and the translation transformation matrix $[c]_k$ are available.[2] Since all 3 matrices are sparce, they have been multiplied and presented in Appendix 1 so that the coefficients S_{ij} may be determined if desired by calculator or minicomputer. Note that many terms become zero if piles are vertical, are symmetric ($I_x = I_y$), or the end condition pinned or free to rotate.

The components of $[b]^{-1}$ may be determined experimentally by load tests as discussed earlier in terms of the foundation stiffness. Although expensive and only true for that pile under site conditions present at the time of the test, the results are tangible and do not require assumptions or hypotheses of modeling. Modeling, however, is the basic tool of analysis and necessary for design. With sufficient confidence in a particular model various pile types and configurations may be tried with only the cost of analysis. Models may be too simple so that results are doubtful and assumptions simply not up to the state of the art. Conversely, sophistication possible in research is academic; for practical design models have to be able to utilize normal data obtained from soil borings and surveys. As a function of risk and cost, this data could range from very limited to extensive. The same arguments apply to field and laboratory soil property tests. Most machines requiring consideration of dynamics in the design of their foundations would be of sufficient importance to warrant at least a boring and some tests. The primary element utilized in the various models presented herein is a linear constant in the relationship between average lateral bearing pressure and deflection, the "beam on a spring foundation" concept. This value may be assumed constant over the length of the pile or constant over an increment of depth or layer. Otherwise, it may be assumed to increase linearly with depth. It is felt that values of this constant, the modulus of subgrade reaction, may be reasonably estimated or measured at present. In addition, progress is being made in improving correlations with test data and other types of "soil modulus" concepts to eventually make it even more reasonable to use this basis for modeling and with an ever better confidence level. Determination of soil properties, especially over a period of time, appears to be the major deterent to a higher confidence level or the use of any greater sophistication. Field tests correlated with soil data will certainly provide the best data eventually. Finally, strength parameters are not sensitive to variation in soil modulus but displacement parameters are; thus the designer can be confident in a force parameter on a pile with an estimated soil modulus but the corresponding deflection may be considerably off. This means that improved soil data is needed for dynamic computations.

The models presented are of 2 classes; a semi-infinite pile or a finite pile. The semi-infinite models may be used whenever either parameter βL or $\psi L > \pi$, where

$$\beta^4 = \frac{k_s D}{4EI} \quad \text{and} \quad \psi^5 = \frac{\eta D}{EI} \tag{5}$$

in which k_s is the modulus of subgrade reaction in units of pressure per unit displacement, i.e., $lb/in.^3$ or kN/mm^3, and $k_s = \eta z$ when the subgrade modulus increases linearly with depth; L is the embedded length of the pile, EI the flexural rigidity of the pile as a beam, and D is the projected width of the pile. There are 2 values of the parameters β and ψ for each pile, one with respect to each principal axis. Validity for use of the semi-infinite model is because of the damped wave form of the elastic curve of the pile which shows that if the pile extends beyond the first wave deflection and all other stress resultants in flexure essentially become negligible. Formulas for coefficients for 4 cases are given in Table 1 where there is a distinction made between a semi-infinite media and a pile extending as a cantilever length ℓ above the semi-infinite media. Subscripts i are adjusted to correspond to the direction or axis of bending so each entry corresponds to two different coefficients unless the pile has the same properties I_i and D_i with respect to both principal axis. In

Table 1 Nonzero Coefficients for Semi-Infinite Piles in Flexure

Model	b_{11} & b_{22}	b_{44} & b_{55}	b_{15} & $-b_{24}$**
A. Single Layer			
1. Beam on a* Constant Spring Fdn.	$(1+\delta)\kappa_i\beta_i^2$ where $\kappa_i = 2\beta_i EI_i$	$\delta\kappa_i$	$\delta\kappa_i\beta_i$
2. Beam on a Linearly Increasing Spring Fdn.	$(0.427+0.672\delta)\lambda_i\psi_i^2$ where $\lambda_i = EI_i\psi_i$	$1.503\delta\lambda_i$	$\delta\lambda_i\psi_i$
B. Two Layer (ℓ is the unsupported length of cantilever)			
1. Cantilever Adjoining Model A1	$3[1+\delta(1+2\beta_i\ell)]\beta_i^2\kappa_i t_i$	$\delta(3+6\beta_i\ell+6\beta_i^2\ell^2$ $+2\beta_i^3\ell^3)\kappa_i t_i$	$3\delta(1+2\beta_i\ell+\beta_i^2\ell^2)\beta_i\kappa_i t_i$
	where $t_i = 1/[3+6\beta_i\ell+6\beta_i^2\ell^2+(1+\delta)2\beta_i^3\ell^3+\delta\beta_i^4\ell^4]$		
2. Cantilever Adjoining Model A2	$3(1+3\delta)[\psi_i+\delta(1.686+\psi_i\ell$ $-\psi_i)]\lambda_i\psi_i^2 p_i$	$4\delta(6.918+9.204\psi_i\ell$ $+5.058\psi_i^2\ell^2+\psi_i^3\ell^3)\lambda_i p_i$	$6\delta(3.068+3.732\psi_i\ell$ $+\psi_i^2\ell^2)\lambda_i\psi_i p_i$
	where $p_i = 1/(18.417\delta+[1+\delta(\ell-1)][6.918(1+3\delta)\psi_i+9.204(1+\delta)\psi_i^2\ell+1.686(3+\delta)\psi_i^3\ell^2+\psi_i^4\ell^3])$		

* Subscripts i must be adjusted for the direction or axis of bending. For the fixed condition $\delta=1$ and for the pinned end $\delta=0$.

** The form of b_{24} is always the same as the form for b_{15} but negative. The values may not be the same however since $\psi_1 \neq \psi_2$ or $\beta_1 \neq \beta_2$ unless piles have $I_x = I_y$ and $D_x = D_y$.

addition, the coefficients b_{15} and b_{24} have the same form but b_{24} is always negative and, although having the same formula, they are only equal when $\beta_1 = \beta_2$ or $\psi_1 = \psi_2$. The 2-layer case is useful to account for a weak surface situation or for an elevated foundation such as a platform. The cantilever beam model for a single layer has been omitted since it is derived using a second unnecessary level of approximation. It can be easily used if desired.

Finite Length Pile or Layered Soil

When βL or $\psi L < \pi$ or the soil-pile system must be modeled in layers because of varying properties of soil strata, nonlinearities in the soil modulus, or the pile is nonprismatic, a finite length beam on a spring foundation model may be used. There may be a large number of layers, but in each segment the pile is assumed to be prismatic and the soil to have a constant modulus of subgrade reaction which may be zero as a lower bound. There is no restriction on the ordering of layers, thus softer strata may underlay stiffer soils. The fourth order differential equation of a prismatic beam on a spring foundation may be solved with 8 undetermined coefficients whose values are determined by stress resultants at the joints[5]. This results in an 8 by 8 member stiffness matrix $[K]_k$ for segment k as shown in Appendix 2, where the local kinematic degrees of freedom are as shown in Fig. 2.

To obtain the stiffness matrix $[b]_k$ as used in Eq. 4 the N by N stiffness matrix for the pile of M finite segments, as shown in Fig. 3, is first formed by adding the stiffness of each segment $[K]_k$ adjusting the kinematic degree of freedom numbers to coincide with the global numbers of Fig. 3. The resulting N by N stiffness matrix [K] is partitioned to isolate the 4 by 4 matrix coinciding with kinematic degree of freedom numbers 1 through 4, i.e., the surface or top end degrees of freedom, and then condensed to obtain a 4 by 4 matrix $[b_f]_k$ which coincides with the flexural degrees of freedom in $[b]_k$. The $[b]_k$ matrix is then formed by adding rows and columns with the axial and torsional coefficients, which are assumed to be uncoupled. Including axial and torsional degrees of freedom in the member of Fig. 2 and the pile of Fig. 3 could be easily done but has been omitted since it would increase the member degrees of freedom to 12 with a corresponding increase in the global degrees of freedom of 50% and, in addition, these actions are uncoupled and can be superimposed. Condensation can be accomplished by Gaussian Elimination, which is preferred in computer programming or by matrix condensation where the partitioned N by N stiffness matrix is written

$$\begin{bmatrix} b_{ff} & b_{fg} \\ b_{gf} & b_{gg} \end{bmatrix}_k \begin{Bmatrix} x_f \\ x_g \end{Bmatrix}_k = \begin{Bmatrix} F_f \\ 0 \end{Bmatrix}_k \tag{6}$$

and therefore

$$[b_f]_k = [b_{ff} - b_{fg}\, b_{gg}^{-1}\, b_{gf}]_k \tag{7}$$

Utilizing any of the several models or from test data the pile stiffness $[b]_k$ is obtained and through use of Eqs. 4 and 2 the stiffness [S] of the foundation.

ANALYSIS

Design of a pile foundation consists of iterative changes and analyses of a preliminary arrangement of piles. Pile type, length, number, spacing, plan angle, and batter angle are all design variables. The vertical component takes precedence because of soil bearing and magnitude of load and helps set the number, type and spacing of piling. Greater spacing is preferred since there is less superposition of soil stresses at depth and potential settlement. Improvement in lateral stiffness may frequently be an objective and is primarily influenced by batter angle γ although increasing the flexural rigidity EI and/or projected width D of the piles helps. Rigidly connecting the pile to the foundation also results in a marked increase in stiffness when compared with a hinged connection. Thus iterative analyses with considered improvements is the design methodology. This can best be accomplished through use of a computer program.*

EXAMPLE

A foundation design was required for the Municipal Power Plant at Larned, Kansas, located in the floodplain of the Arkansas River. The decision was made to use piling as the best solution to a problem brought about by a poorly compacted deep silt soil condition and a fluctuating water table. The installation provides for flooding. Data concerning the installation are given in Table 2 and Figs. 4 and 5. Use of piling improved bearing capacity and provided lateral stiffness.

The soil conditions at the power plant site consisted of a silty clay to a depth of about 10 feet underlain by fine sand. Insufficient information was provided to determine the modulus of subgrade reaction and so estimates were made. Although several soil-pile interaction models were

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* A program of about 600 FORTRAN statements was written for complete analysis of a general pile foundation allowing a choice of 5 pile models allowing computation of stiffness and for any kind of loading. Forces and deflections of individual piling are obtained.

Table 2. Foundation Data for Example Problem

Manufacturer Colt Industries, Fairbanks Morse Engine Division, Beloit, Wisconsin.

Model & Engine Data Colt-Pielstick PC-2 18 cylinder V dual fuel diesel 9000 hp. operating at 514 rpm. See photograph, Fig. 4.

Dimensions and Weight (see Fig. 5)

Item	Description	Weight	Dimensions x	y	z
A	Engine	192.7 k	320"	80"	80"
B	Alternator	77.0	50	100	100
C	Exciter	5.7	20	120	30
D	Foundation	704.5	544	132	129
E	In phase soil	274.3	544	132	60

Mass	Engine Unit & Foundation	Engine Unit, Fdn. & 5' of soil
mass	30.5 k-sec^2/ft	39.0
I_x	1230 k-ft-sec^2	2100
I_y	5250	7560
I_z	4600	6220

Soil Poorly consolidated; silty clay to about 10 ft underlain by a fine sand with a fluctuating water level. Soil data poor and it was therefore assumed that k_s(kci) $= \sqrt{0.001z}$ where z is in ft.

Table 2 (cont.)

Computed Frequencies (radians/sec)

a) Engine Unit & Foundation

pile batter	ω_x	ω_y	ω_{zz}	ω_{xx}	ω_{yy}	ω_z
Vertical	24.0	24.0	27.2	77.9	159.8	161.7
5	26.0	28.1	35.9	79.3	157.5	159.5
4	26.8	28.7	39.7	80.4	156.3	158.3
3	28.1	29.5	46.3	82.7	153.8	155.9
2	30.2	31.2	59.5	88.4	147.4	149.8

b) Engine Unit, Foundation & 5' of Soil

pile batter	ω_x	ω_y	ω_{zz}	ω_{xx}	ω_{yy}	ω_z
Vertical	21.3	21.6	23.5	64.5	136.6	143.0
5	23.0	24.7	31.1	65.9	134.6	141.0
4	23.7	25.2	34.3	67.1	133.6	139.9
3	24.9	25.7	40.1	69.4	131.4	137.8
2	25.9	27.6	51.5	75.2	126.0	132.4

Pile Data

24-12 BP 53 piles of 30 to 35 ft were driven to a computed capacity of 70 to 90 kips in 3 rows. Piles in the perimeter were battered alternately at 1 to 3 or 1 to 4. Computations herein are based on an earlier trial with 33 piles in 3 rows with all perimeter piles battered, those at the corners at 45°. The piles in the study are 10" XS ϕ pipe spaced at 50" in the longitudinal direction and the rows 46" apart, all dimensions center-to-center of piles.

examined the one used in this report is an assumed parabolic increase with depth from zero at the base of the foundation. Such a variation seems reasonable because it attributes a low stiffness to the relatively unconfined surface layers and an increasing stiffness in the underlying sand. Broms[6] suggests that the dynamic modulus be taken as a fraction of the soil reaction modulus for static loading, recognizing the softening effect of repetitive loading of the soil.

Mass included the machine and foundation block with all appurtenances and pockets. Computations for frequency were made with and without added soil mass. Since the inflection point for the fixed-head pile is at roughly $3/\beta$ depth, about a third or 60 in. of this was used to compute added mass. The resulting lower frequencies with added mass are given in Table 2. The end or boundary condition of a fixed-head is obvious with a steel pile embedded in concrete. If the pile were hinged, that is, resistance to moment negligible, the stiffness and therefore frequency would be considerably decreased, in the example by a third to over a half.

Batter, or the slope of the pile as a ratio of vertical to horizontal projection, h, where $h = \cot\gamma$, directly affects stiffness and therefore frequency. In Table 2 the results of trials with 4 different batter slopes and vertical are reported. The reported frequencies correspond to modes denoted by subscripts, the single subscript being translational in the direction indicated and as shown in Fig. 5 and the rotational by double subscripts about the axes indicated. Since the system is not symmetric, the center of mass does not coincide with the geometric centroid of the pile group, or center of stiffness, there is coupling in the modes not indicated in Table 2. This was most noticeable in the rotational modes about the horizontal axes, ω_{xx} and ω_{yy}, where translational and torsional components were also present.

The 2 translational modes in the horizontal plane, ω_x and ω_y, and the torsional mode, ω_{zz}, all rely on lateral stiffness. There is also a strong element of this in the rotational mode, ω_{xx}, because of coupling. When all piling are vertical this stiffness is supplied solely by the flexural rigidity of the piles. As the batter angle γ is increased the component of axial rigidity of the pile participating in the horizontal plane or adding to lateral stiffness increases, adding to the flexural contribution and therefore increasing the frequency in these modes. There is a corresponding decrease in frequency in modes, ω_{yy} and ω_z, relying on pile axial stiffness.

Computations were made using the computer program referred to earlier to obtain stiffness and another program[7] to compute frequencies and modes. Values of pile forces under operating conditions were evaluated. Due to steady state operating conditions damping was not considered since resonance was avoided and dynamic forces were slightly overestimated or on the safe side. The installation has been operating since 1976.

CONCLUSIONS

Utilization of piling in machine foundations is shown to provide more flexibility to the designer and quite possibly result in a more economical design for some cases. It can also provide a solution to some difficult foundation problems. The key for useful utilization is a complete analysis method which provides for a choice of practical models for simulating the lateral resistance of piles as well as their axial and torsional behavior. The formulation is very general allowing location of the pile heads at different elevations, any spacing, plane angle, or batter of piles, or a mixture of pile types or models. Solution of Eq. 1 yields frequencies which may be sufficient data for a steady state operating machine such as the example. Numerical integration may be necessary in other cases, but the required parameters are provided.

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Appendix 1

Formulas for Stiffness Influence Coefficients

Formulas are given for single piles. Stiffness coefficient $S'_{ij} = S'_{ji}$ by reciprocity and function B_i are defined for convenience as follows:

$B_1 = b_{11} \cos^2\gamma - b_{22} + b_{33} \sin^2\gamma$

$B_2 = (b_{11}-b_{33}) \sin\gamma \cos\gamma$

$B_3 = (b_{15}+b_{24}) \cos\gamma \sin\alpha \cos\alpha$

$B_4 = u_1 \sin\alpha - u_2 \cos\alpha$

$B_5 = b_{11} \sin^2\gamma + b_{33} \cos^2\gamma$

$B_6 = b_{44} \cos^2\gamma - b_{55} + b_{66} \sin^2\gamma$

$B_7 = u_3 (b_{22}+B_1 \cos^2\alpha)$

$B_8 = u_3 (b_{22}+ B_1 \sin^2\alpha)$

$B_9 = B_1 u_3 \sin\alpha \cos\alpha$

$B_{10} = B_1 \cos\alpha$

$B_{11} = B_2 \cos\alpha$

$B_{12} = B_1 \sin\alpha$

$B_{13} = B_2 \sin\alpha$

$B_{14} = (b_{15} \cos^2\alpha - b_{24} \sin\alpha)\cos\alpha$

$B_{15} = b_{15} \sin\alpha$

$B_{16} = (b_{15} \sin^2\alpha - b_{24} \cos^2\alpha)\cos\gamma$

$B_{17} = u_1 b_{22} - b_{24} \sin\gamma \cos\alpha$

$B_{18} = \sin\alpha \cos\alpha$

$B_{19} = (b_{44} - b_{66}) \sin\gamma \cos\gamma$

Thus,

$S'_{11} = B_{10} \cos\alpha + b_{22}$

$S'_{12} = B_1 B_{18}$

$S'_{13} = -B_{11}$

$S'_{14} = -u_2\ B_{11} - B_9 - B_3$

$S'_{15} = u_1\ B_{11} + B_7 + B_{14}$

$S'_{16} = B_4\ B_{10} - u_2 b_{22} + b_{24}\ \sin\gamma\ \sin\alpha$

$S'_{22} = B_{12}\ \sin\alpha + b_{22}$

$S'_{23} = -B_{13}$

$S'_{24} = -u_2\ B_{13} - B_8 - B_{16}$

$S'_{25} = B_3 + B_9 + u_1 B_{13}$

$S'_{26} = B_4\ B_{12} + B_{17}$

$S'_{33} = B_5$

$S'_{34} = u_2 B_5 + u_3 B_{13} + B_{15}\ \sin\alpha$

$S'_{35} = -u_1 B_5 - u_3 B_{11} - B_{15}\ \cos\alpha$

$S'_{36} = -B_2\ B_4$

$S'_{44} = u_2^2 B_5 + 2u_2 B_{15}\ \sin\alpha + B_6\ \cos^2\alpha + b_{55} + u_3(2u_2 B_{13} + B_8 + 2B_{16})$

$S'_{45} = -u_1 u_2 B_5 + B_6 B_{18} - B_{15}(u_1\ \sin\alpha + u_2\ \cos\alpha) - u_3(u_2 B_{11} + u_1 B_{13} + u_3 B_1 B_{18} + 2B_3)$

$S'_{46} = u_2(B_3 - B_2\ B_4) - u_1 B_{16} - B_{19}\ \cos\alpha - u_3(B_{12}\ B_4 + B_{17})$

$S'_{55} = u_1^2 B_5 + B_6\ \sin^2\alpha + b_{55} + 2u_1 B_{15}\ \cos\alpha + u_3(B_7 + 2u_1 B_{11} + 2B_{14})$

$S'_{56} = u_1(B_2\ B_4 + B_3) - B_{19}\ \sin\alpha - u_2 B_{14} + u_3\ [\sin\gamma(u_1 B_1 \cos\gamma + b_{24}\ \sin\alpha) - u_2(B_{10}\ \cos\alpha + b_{33})]$

$S'_{66} = B_1 B_4^2 + (u_1^2 + u_2^2) b_{22} - 2b_{24}(u_1 \cos\alpha + u_2 \sin\alpha)\sin\gamma + (b_{44} - b_{66})\sin^2\gamma + b_{66}$

Where $U_i(u_1, u_2, u_3)$ are the coordinates of the pile top in the foundation, α_i is the angle to the direction of batter measured clockwise in plan from the U_1 axis and γ_i is the angle of batter from the vertical in the plane of batter.

Appendix 2

Stiffness Matrix for a Pile Segment

Refer to Fig. 2

$$
[K]_k = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{bmatrix}
T3_y & 0 & 0 & T5_y & T4_y & 0 & 0 & -T6_y \\
0 & T3_x & -T5_x & 0 & 0 & T4_x & T6_x & 0 \\
0 & -T5_x & T1_x & 0 & 0 & -T6_x & T2_x & 0 \\
T5_y & 0 & 0 & T1_y & T6_y & 0 & 0 & -T2_y \\
T4_y & 0 & 0 & T6_y & T3_y & 0 & 0 & -T5_y \\
0 & T4_x & -T6_x & 0 & 0 & T3_x & T5_x & 0 \\
0 & T6_x & T2_x & 0 & 0 & T5_x & T1_x & 0 \\
-T6_y & 0 & 0 & -T2_y & -T5_y & 0 & 0 & T1_y
\end{bmatrix}_k
\end{array}
$$

where,

$$T1_i = (C'S' - CS)\kappa q$$

$$T2_i = (C'S - CS')\kappa q$$

$$T3_i = 2(CS + C'S')\kappa\beta^2 q$$

$$T4_i = 2(C'S + CS')\kappa\beta^2 q$$

$$T5_i = (S'^2 + S^2)\kappa\beta q$$

$$T6_i = 2SS'\kappa\beta q$$

$$q = 1/(S'^2 - S^2)$$

$$C = \cos\beta L$$

$$S = \sin\beta L$$

$$C' = \cosh \beta L$$

$$S' = \sinh \beta L$$

$$\kappa = 2\beta EI$$

$$\beta^4 = k_s D/(4EI)$$

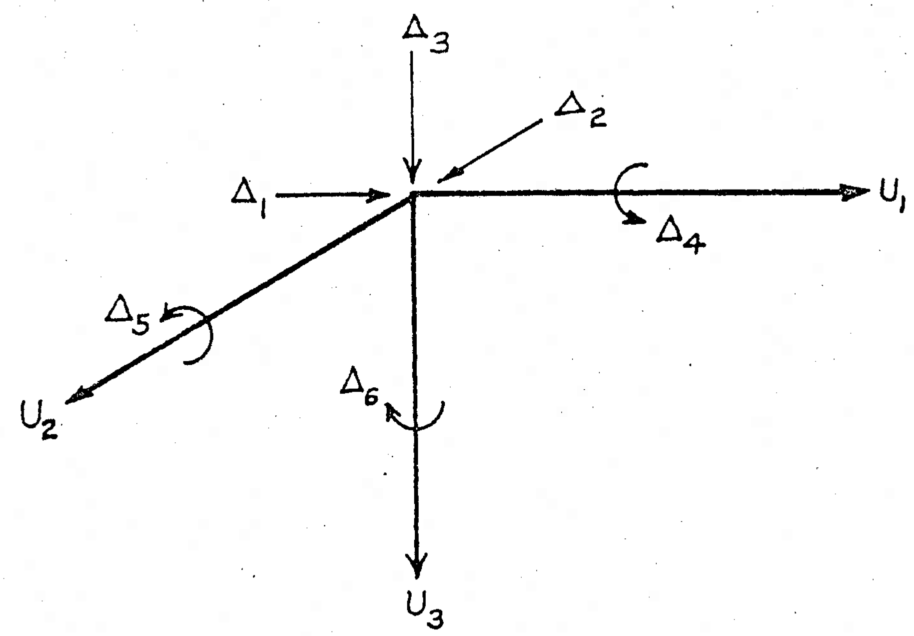

Figure 1. Coordinate System U_i and Kinematic Degrees of Freedom Δ_i.

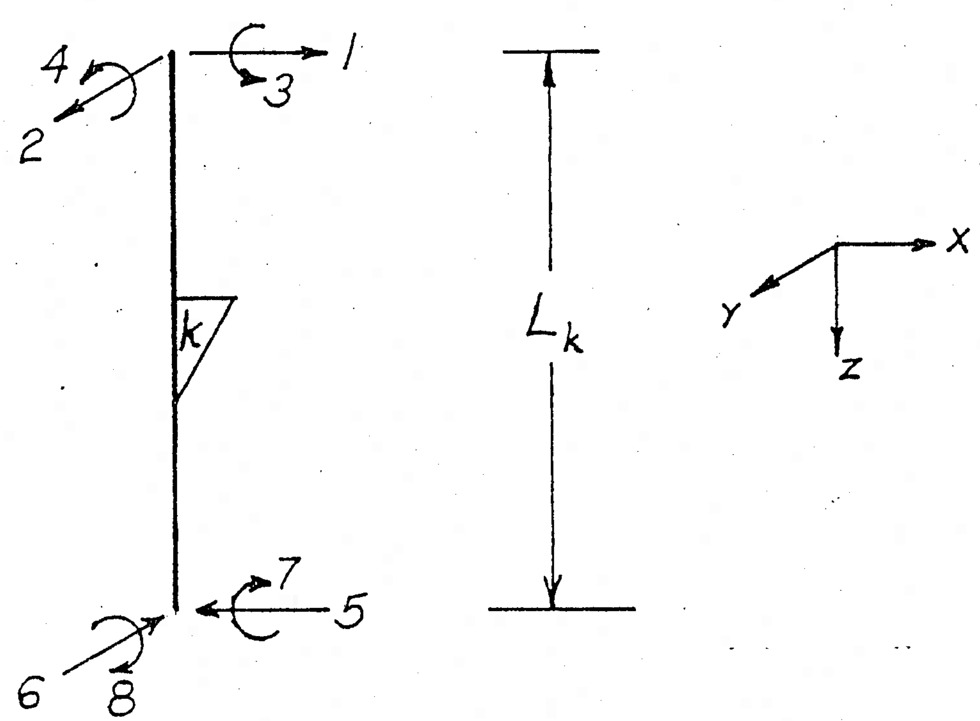

Figure 2. Segment k of Pile. Kinematic Degrees of Freedom are Local.

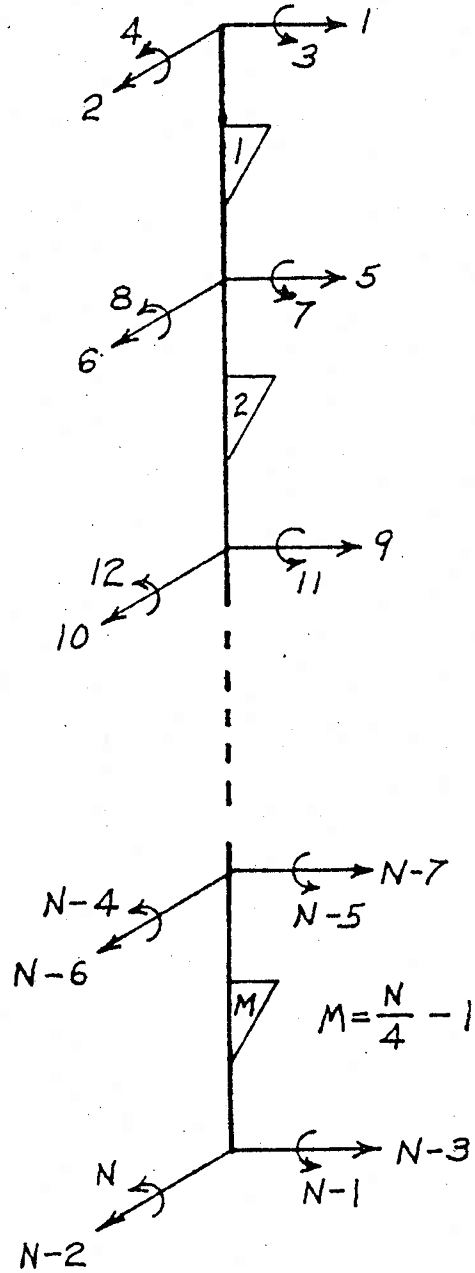

Figure 3. Model of Pile of M Finite Segments

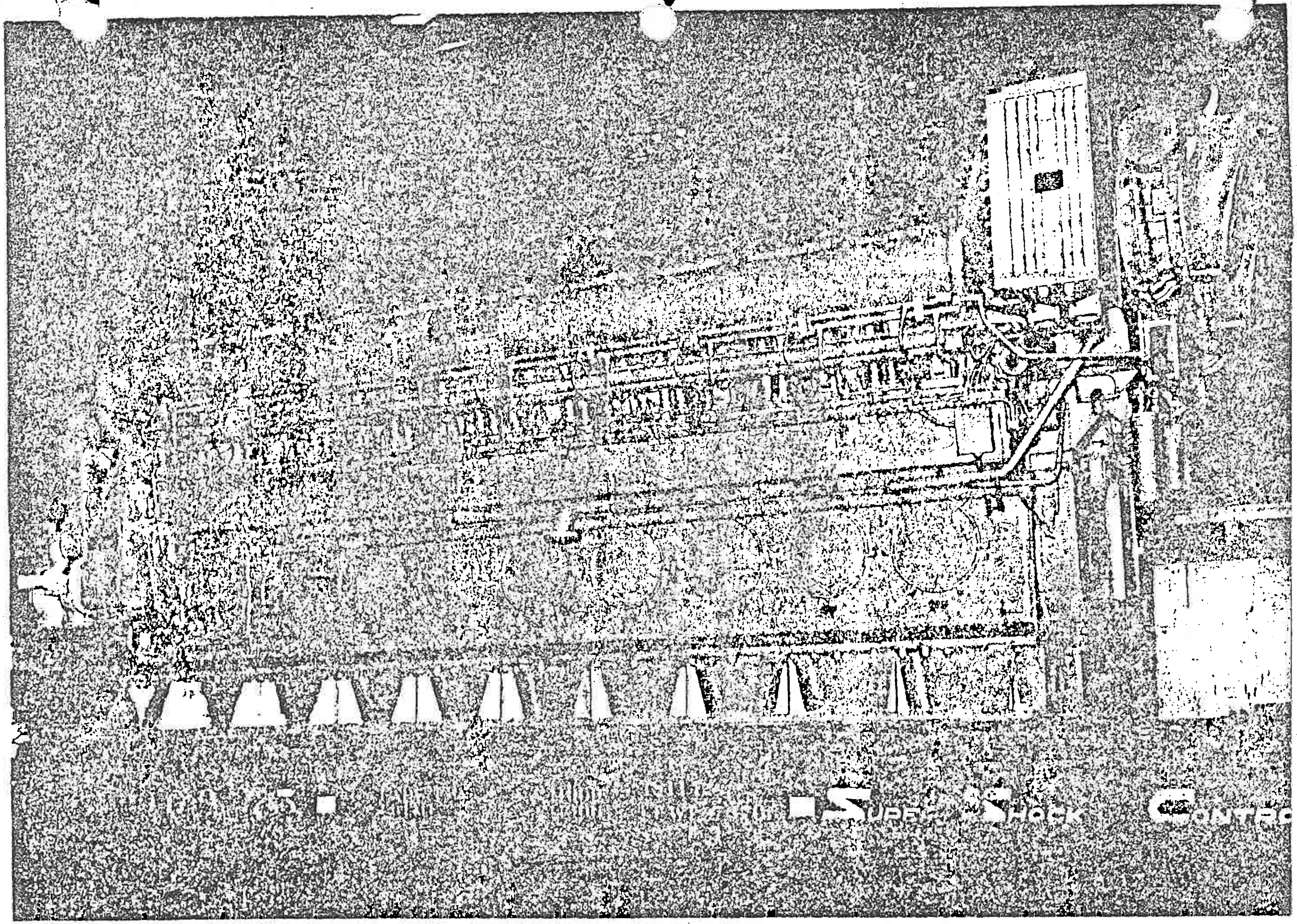

Figure 4. Photograph of Colt-Pielstick V 18 Diesel Installed for Power Plant on Pile-Supported Foundation at Larned, Kansas

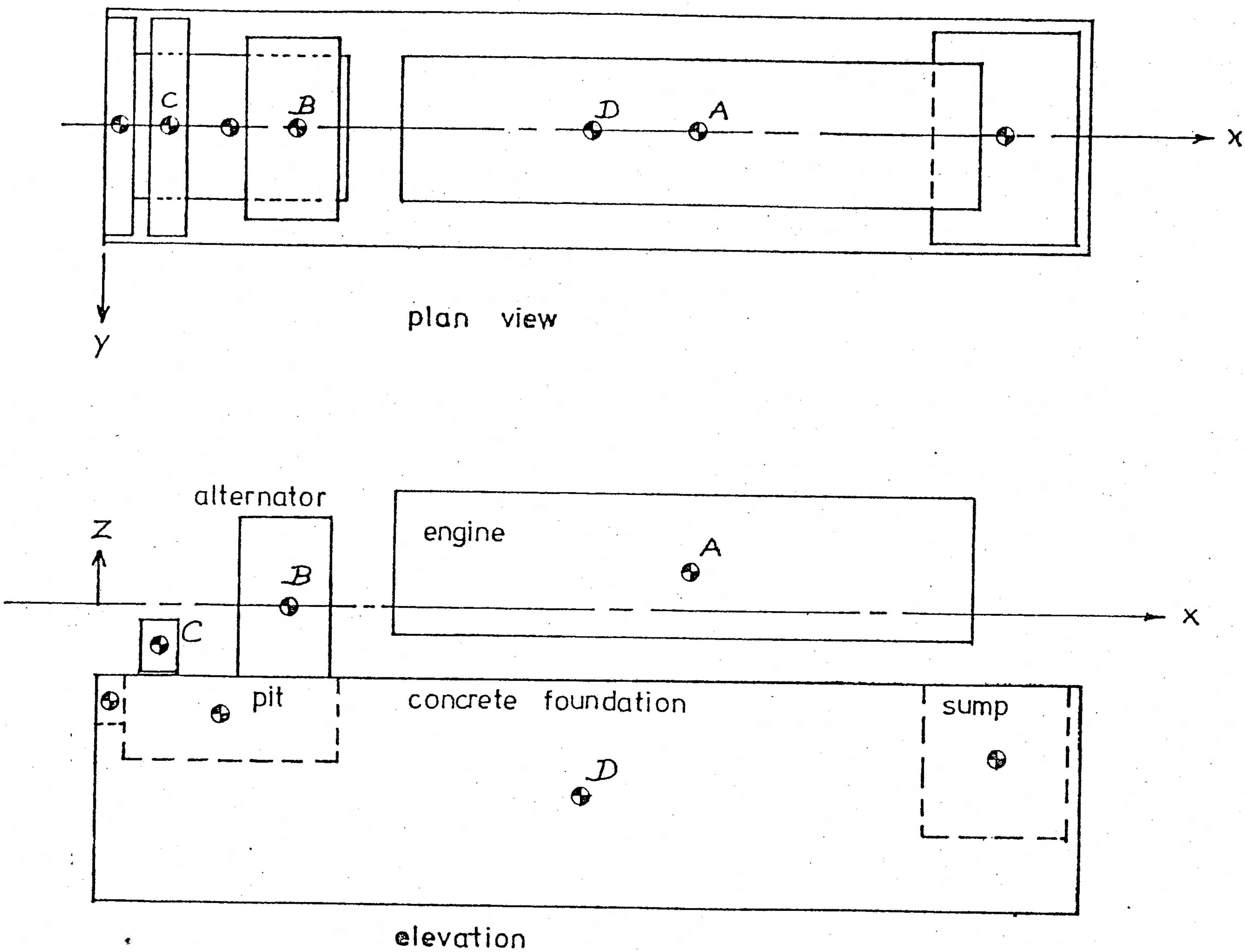

Figure 5. Foundation & Machinery

NOTES ON PILE STIFFNESS

I. Axial

1. H.G. Poulos: [see "Load-Settlement Prediction for Piles and Piers", H.G. Poulos, ASCE Journal SM9, Sept. 72, pp. 879-895]. Based on theory of elasticity, formulation as a rigid body in a semi-infinite mass, extended for compressible pile, later for nonhomogeneous soil. Used to predict load-settlement curve to failure

$$b_{33} = \frac{P_{yi}}{\rho_{yi}} = \frac{E_s d}{I} \qquad \text{below "yield"}$$

E_s = modulus of elasticity of soil

d = pile diameter

I = pile setttttlement influence factor

$= I_1 R_k R_h R_b$ where each variable is obtained from a prepared curve.

In a later paper ["Settlement of Pile Foundations," by H. G. Poulos Numerical Methods in Geotechnical Engineering, ed. C.S. Desai & J. T. Christian, McGraw-Hill, 1977, pp. 326-363], Poulos gives b_{33} in a different form

$$b_{33} = \frac{P}{\rho} = \frac{LE_s}{I_p}$$

where L is pile length, E_s the soil modulus, and I_p an influence settlement factor. See Fig. 10.6 of paper (p. 347) for chart to determine $I_p(K,\ L/D)$ for a floating pile. There are other charts for corrections for end bearing, group action, etc.

2. M. Novak: [see "Dynamic Stiffness and Damping of Piles" M. Novak, Can. Geotech. J., Vol. 11, 1974, pp. 574-598]. Based on a model which assumes soil to be composed of

independent infinitesimally thin horizontal layers that extend to infinity, a generalized beam-on-spring-foundation model. Pile may be short but is assumed hinged at tip, i.e., tip does not move vertically

$$b_{33} = \frac{E_p A}{r_o} f_{18,1} \qquad f_{18,1} = \frac{F_{18}(\Lambda)_1}{\ell/r_o}$$

where E_p = modulus of elasticity of pile

A = area of pile

r_o = radius of pile

$f_{18,1} = f_{18,1}\ (V_s/v_c,\ \ell/r_o)$ are given in Fig. 9 of paper

$V_s = \sqrt{G/\rho}$ = shear wave velocity of soil

G = shear modulus of elasticity of soil

$\rho = \frac{w}{g}$ = mass density of soil (e.g., $\frac{\text{lb-sec}^2}{\text{ft}^4}$

$v_c = \sqrt{E_p/\rho_p}$ = longitudinal wave velocity in pile

ρ_p = mass density of pile

$F_{18}(\Lambda) = \Lambda \cot \Lambda = F_{18}(\Lambda)_1 + i\, F_{18}(\Lambda)_2$, a complex function $F_{18}(\Lambda)_1$ is the real part

$$\Lambda = \ell \sqrt{\frac{1}{E_p A}[\mu\omega^2 - GS_{\omega 1} - i(c\,\omega + GS_{\omega 2})]}$$

in which ω = excitation frequency

μ = mass of pile per unit length

c = coef. of pile internal damping

$S_{\omega 1}$ & $S_{\omega 2}$ are Bessel functions

For static conditions $\omega = 0$ leads to $\Lambda = L\sqrt{\frac{4\pi r G}{E_p A}}$

$$f_{18,1} = \frac{r_o}{L} \Lambda \coth \Lambda$$

3. A.S. Vesić [see "Design of Pile Foundatins", A.S. Vesić, TRB Synthesis #42, 1977 (pp. 22-43 for material of interest here)].

$$x_3 = w_s + w_{pp} + w_{ps}$$

where

$w_s = (Q_p + \alpha_s Q_s)\frac{L}{A_p E_p}$ deformation of pile

$w_{pp} = \frac{q_p B}{E_s^*} I_{pp} = \frac{C_p Q_p}{B q_o}$ pile point settlement due to point load

$w_{ps} = \frac{\bar{f}_s B}{E_s^*} I_{ps} = \frac{C_s Q_s}{D q_o}$ pile point settlement due to shear

Q_p = point load = βQ_t

Q_s = shear load = $(1-\beta) Q_t$

$Q_p + Q_s = Q_t$ = axial load on pile

$\alpha_s = \alpha_s$(diam., distribution of skin friction)

$\frac{1}{10} < \alpha_s < 2/3$ (approx.) use $\alpha_s = 1/2$ for uniform skin friction & prismatic pile.

L, A_p, E_p = pile properties

I_{pp} & I_{ps} are influence coefs. from theory of elasticity (see Poulos).

B = diam. of pile

D = embedded length of pile

$E_s^* = E_s/(1-\nu_s^2)$ = plane strain modulus of elasticty of soil, ν_s = Poisson's ratio

q_p = net pressure on pile point

$\bar{f}_s$ = average shear on pile surface (skin friction)

C_p & C_s are empirical coefficients where C_p comes from Table 6 and

$$C_s = (0.93 + 0.16\sqrt{D/B})\, C_p$$

q_o = ultimate point resistance.

Therefore,

$$\frac{1}{b_{33}} = [\beta + \sigma_s(1-\beta)]\frac{L}{A_p E_p} + \frac{C_p \beta}{B q_o} + \frac{C_s(1-\beta)}{D q_o}$$

4. M.F. Randolph & C.P. Wroth: [see "Analysis of Deformation of Vertically Loaded Piles," M.F. Randolph & C.P. Wroth, ASCE Jrl. GT12, Dec. 1978, pp. 1465-1487]. An approximate analytic expression.

$$b_{33} = \frac{P_t}{w_t} = G_\ell r_o \left[\frac{4}{\eta(1-\nu)} + \frac{2\pi}{\xi}\rho\frac{\ell}{r_o}\frac{\tanh(\mu\ell)}{\mu\ell}\right]\left[1 + \frac{4}{\eta(1-\nu)}\frac{1}{\pi\lambda}\frac{\ell}{r_o}\frac{\tanh(\mu\ell)}{\mu\ell}\right]^{-1}$$

where

$$\xi = \ln(r_m/r_o) = \ln[2.5(\ell/r_o)\rho(1-\nu)]$$

$$\mu\ell = \sqrt{2/(\xi\lambda)}\,(\ell/r_o)$$

$\rho = G(\ell/2)/G(\ell)$ (ratio of G at midheight to G at tip)

$\eta = 1$

G = soil modulus

r_o = pile radius

ν = Poisson's ratio of soil

$\lambda = E_p/G_s$

ℓ = pile length, E_p = pile modulus

Notes on Pile Stiffness

5. Examples:

Given: Pile; Solid cylinder, concrete, $E_p = 3000$ ksi, $L = 25$ ft, $D = 1$ ft. soil; loose sand, $E_s = 3.5$ ksi, $\nu = 0.4$; $\gamma = 110$ pcf, uniform.

(a) Poulos.

$L/d = 25 \Rightarrow$ Fig. 1 $\Rightarrow I_1 \approx 0.075$

$$K = \frac{E_p}{E_s} R_A = \frac{3000}{3.5}(1) \qquad \text{Solid Pile, } R_A = 1$$

$$= 860 \qquad \text{Fig. 3} \Rightarrow R_k \doteq 1.2$$

$R_h = 1$ for pile in a semi-infinite mass

$R_b = 1$ for a uniform soil

$\therefore \; I = I_1 R_k R_h R_b = 0.075(1.2)(1)(1) = 0.09$

Ratio of load carried by base

$\beta = \beta_1 C_k C_b = 0.057(0.92)(1) = 0.0525$

$\beta_1 \Rightarrow$ Fig. 2 , $C_k \Rightarrow$ Fig. 4

(not needed here, used later)

$$b_{33} = \frac{E_s d}{I} = \frac{3.5(12)}{0.09} = 467\text{k/in.}$$

or, using Fig. 10-6 of 2d paper: $b_{33} = \frac{E_s L}{I_p} = \frac{3.5 \times 25 \times 12}{2.2} =$ 477k/in.

(b) Novak.

$\ell/r_o = \frac{25}{.5} = 50$

$$\rho_s = \frac{110}{32.2} = 3.4161 \; \frac{\text{lb-sec}^2}{\text{ft}^4}$$

$$\rho_p = \frac{145}{32.2} = 4.5031 \quad "$$

$$G = \frac{E_s}{2(1+\nu)} = \frac{3.5}{2.8} = 1.25 \text{ ksi}$$

$$V_s = \sqrt{\frac{G}{\rho_s}}$$

$$= \frac{1.25(1000)(144)}{3.4161} = 230 \text{ ft/sec} = 2755 \text{ in/sec}$$

Notes on Pile Stiffness

$$v_c = \sqrt{E_p/\rho_p} = \sqrt{\frac{3000(1000)(144)}{4.5031}} = 9795 \text{ ft/sec}$$

$$\frac{V_s}{V_c} = \frac{230}{9795} = 0.0234$$

$\therefore$ Fig. 9 $\Rightarrow$ $f_{18,1} = .025$

$$b_{33} = \frac{E_p A}{r_o} f_{18,1} = \frac{3000(\pi)(6)^2}{6} (.025) = 1414 \text{ k/in}$$

$$\text{or, } \Lambda = L\sqrt{\frac{4\pi r G}{E_p A}} = 25\text{x}12\sqrt{\frac{1.25(4\pi)6}{3000(\pi)6^2}} = 5, \quad \Lambda \coth \Lambda = 5.00045$$

$$b_{33} = \frac{E_p A}{r_o} \frac{F_{18}}{L/r_o} = \frac{E_p A}{L} F_{18} = \frac{5E_p A}{L} = 5654 \text{ k/in.}$$

(c) Vesić.

from p 5 $\beta = 0.0525$; from Eq. 15 Randolph & Wroth

$$\beta = \left[1 + \frac{\pi\eta(1-\nu)}{2\xi}\frac{\ell}{r_o}\right]^{-1}$$

$$\xi = \ln[2.5(\frac{\ell}{r_o})\rho(1-\nu)] = \ln[2.5(50)(1)(.6)] = 4.32$$

$$\beta = [1 + \frac{\pi(1)(.6)}{2(4.32)}(50)]^{-1} = 0.084$$

thus, $0.05 < \beta < 0.08$, will use $\beta = 0.08$

$$q_o = cN_c + q_v N_q = 0(N_c) + .11(25)(20) \qquad N_q \text{ est. @ } 20$$
$$= 55 \text{ k/ft}^2 = .382 \text{ k/in}^2$$

Table 6 $\Rightarrow$ $C_p \simeq 0.04$ $\qquad$ α_s for sand = 2/3

$$C_s = (0.93 + 0.16\sqrt{\frac{25}{1}})\ 0.04 = 0.069$$

$$b_{33} = \left[(\beta+\alpha_s(1-\beta))\frac{L}{A_p E_p} + \frac{C_p \beta}{Bq_o} + \frac{C_s(1-\beta)}{D}\right]^{-1}$$

$$= \left[(.08 + \frac{2}{3}(1-.08))\frac{2.5(12)}{\pi 6^2(3000)} + \frac{0.04(.08)}{12(.382)} + \frac{0.069(1-.08)}{25(12)(.382)}\right]^{-1}$$

$$= 536 \text{ k/in.}$$

(d) Randolph & Wroth

$\xi = 4.32$ (see above) $\qquad \lambda = \frac{E_p}{G_o} = \frac{3000}{1.25} = 2400$

$$\mu\ell = \sqrt{\frac{2}{\xi\lambda}}\,\frac{\ell}{r_o} = \sqrt{\frac{2}{4.30(2400)}}\,(50) = .694$$

$\tanh(\mu\ell) = 0.6010$

$$b_{33} = 1.25(6)\left[\frac{4}{.6} + \frac{2\pi(1)(50)(.6010)}{4.3175(.694)}\right]\left[1 + \frac{4}{.6}\,\frac{1(50)(.6010)}{(2400)(.694)}\right]^{-1}$$

$= 503$ k/in.

(e) Summary

$$b_{33} = k_L\left(\frac{AE}{L}\right) = \frac{3000(\pi 6^2)}{25(12)}k_L = 1131\ k_L$$

Method	b_{33}	k_L
Poulos	467-477 k/in.	0.41-0.42
Novak*	1414-5654	1.25-5.00
Vesic	536	0.47
Ran & Wr	503	0.44
Saul(68)*	1131	1.00

*Both in error, assumes no pile tip displacement.

II. Torsion

1. M.W. O'Neill: [see Discussion in ASCE ST2 Feb. 1969]

$$b_{55} = \sqrt{4\pi r^2 G_s G_p J}$$

wgere r = pile radius

G_s = Soil shear modulus of elasticity

G_p = Pile " " " "

J = torsional constant of pile

Derived from strength of materials approach assuming elastic pile & soil.

2. H.G.Poulos: [Ref. "Torsional Response of Piles," H.G. Poulos, ASCE Jrl., GT10, Oct. 75, pp. 1019-1035] Used integration of equations based on theory of elasticity

$$b_{66} = G_s d^3 \frac{F_\phi}{I_\phi}$$

where G_s = soil modulus

d = pile diameter

$I_\phi = I_\phi(K_T, L/d)$ = influence coefficient

L = pile length

$$K_T = \frac{G_p J_p}{G_s d^4}$$

J_p = pile torsional constant

G_p = pile shear modulus of elasticity

$F_\phi = F_\phi(\frac{T}{T_u}, K_{Te})$, $K_{Te} = K_T(25/L/d)^2$

$0. < F_\phi < 1.0$ soil slip factor

3. M. Novak & J.F. Howell: [Ref. "Torsional Vibration of Pile Foundations," Novak & Howell, ASCE Jrl., GT4, Apr. 77, pp. 271-285]. Derivation similar to I2.

fixed tip $b_{66} = \frac{G_p J}{r_o} f_{\tau,1}$, $f_{\tau,1} = \frac{F_\tau(\Lambda)_i}{\ell/r_o}$ see Fig. 5

pinned tip substitute $F_\tau(\Lambda)_1$ for $F_\tau(\Lambda)_1$, no curve given

May be written $b_{66} = \frac{G_p J}{L} \omega L \coth \omega L$, $\omega = \sqrt{\frac{4\pi r^2 G_s}{G_p J}}$

4. Torsion examples

See Section I part 5 of these notes for data.

$$J = \frac{\pi d^4}{32}$$ (polar moment of inertia for circle).

a) O'Neill

$$G_p = \frac{E_p}{2(1+\nu)} = \frac{3000}{2(1+.25)} = 1200 \text{ ksi}$$

$$J = \frac{\pi(12)^4}{32} = 2036 \text{ in.}^4$$

Notes on Pile Stiffness

$$b_{66} = \sqrt{4\pi (6)^2 (1.25)(1200)(2036)} = 37{,}168 \text{ in-k/radian}$$

(b) Poulos

$$K_T = \frac{G_p J_p}{G_s d^4} = \frac{1200(2036)}{1.25(12)^4} = 94.26 \quad , \quad L/d = 25$$

Fig. 4 $\Rightarrow$ $I_\phi = 0.052$, F_ϕ assumed 1 (no slip)

$$b_{66} = G_s d^3 \frac{F_\phi}{I_\phi} = 1.25(12)^3 \frac{1}{0.052} = 41{,}536 \text{ in-k/radian}$$

c) Novak and Howell

$$\frac{V_s}{V_c} = 0.0234 \text{ (Sect. I, Pt. 5b)}$$

$$a_o = r_o \omega \sqrt{\rho/G} \quad \text{dimensionless frequency}$$

if $\omega = 0$ (static condition)

$$\omega = \sqrt{\frac{4\pi r^2 G_s}{G_p J}} = \sqrt{\frac{4\pi (6)^2 1.25}{203.6(1200)}} + 0.0152$$

$\omega L = 4.564$ $\quad$ $\coth(\omega L) = 1.0002$ $\quad$ $K_T = 4.565$

$$b_{66} = \frac{G_o J \omega L \coth(\omega L)}{L} = \frac{1200(2036)(4.565)}{25 \times 12} = 37{,}170 \text{ in-k/Rad}$$

or, $\sqrt{\frac{G_s}{2G_p}} = \sqrt{\frac{1.25}{2(1200)}} = 0.0228 \Rightarrow$ Fig. 6 yields

$$f_{\tau,1} = 0.0911 \quad \therefore \quad b_{66} = \frac{G_p J}{r_o} f_{\tau,1} = \frac{1200(2036)(0.0911)}{6}$$

$$= 37{,}100 \text{ in-k/Rad}$$

d) Summary

$$b_{66} = K_T \frac{JG}{L} = \frac{2036(1200)}{25(12)} K_T = 8140\ K_T$$

Method	b_{66}	K_T	
O'Neill	37,200	4.57	
Poulos	41,500	5.10	
Novak & H.	37,100	4.56	
Saul	8,140	1.00	assumed