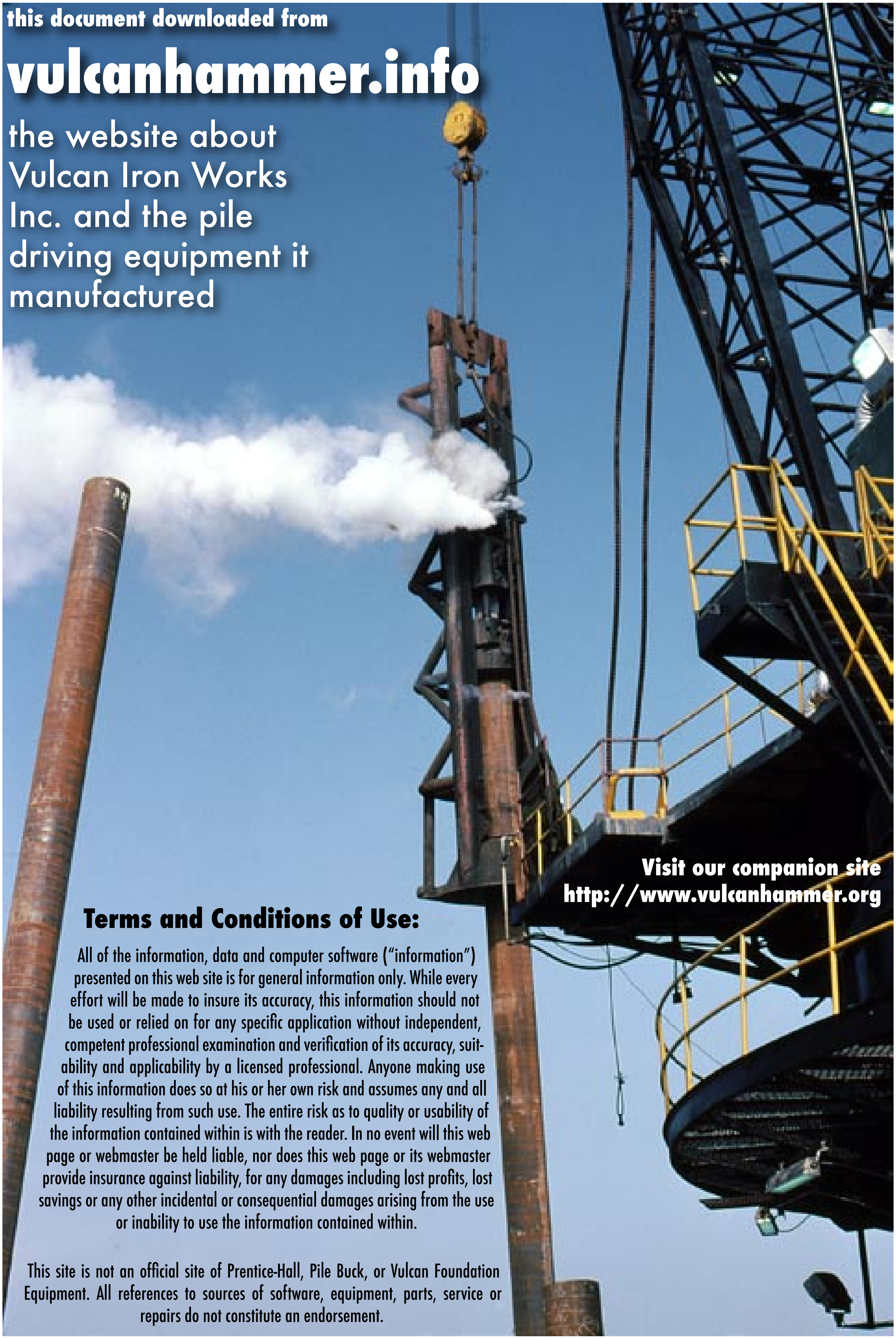

# A Short Introduction to Continuous and Discrete Wave Mechanics

by

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## 1. Introduction

In this discussion of impact problems the wave mechanics concepts of both continuous and discrete pile models are illustrated. The equations which will be derived should be helpful in the understanding of the pile driving process, the Case Method, CAPWAP, WEAP, integrity testing and data interpretation. Examples and some useful numerical values will be given. Always a uniform pile with linear properties will be assumed except where otherwise noted. Also, when a length $\Delta L$ is considered it should be understood that $\Delta L$ is a short distance; $\Delta t$ is considered a short time increment.

## 2. Linear Wave Mechanics

### 2.1 Proportionality

When a rod is struck at the end by a mass a small zone of material is first compressed (Figure 1). This compression causes a strain $\varepsilon$ and therefore a force $F = \varepsilon A E$ (A is the cross sectional area and E is Young's Modulus of the rod). F then compresses a neighboring particle. However, since material is compressed a motion of the particles is also necessary. We speak of a particle velocity, v, in the rod. Whenever a velocity is given to a particle of mass, m, within a time period $\Delta t$ the particle has to be accelerated causing an inertia force $(v/\Delta t)m$. This inertia force is in balance with the strain force and because it takes time to accelerate the particles the strain will be transferred at a certain speed, c, called the wave speed (in/s or ft/s).

Consider Figure 1. The wave has already traveled to a point which is still at rest. During a time interval $\Delta t$ (sec) the wave travels a distance $\Delta L = (\Delta t)c$ (ft) and since the material below the point became compressed, the point moved a distance $\partial$.

This deformation $\partial$ was caused by a strain $\varepsilon$ over a distance $\Delta L$ and therefore

$$\varepsilon = \partial/\Delta L$$

But replacing $\Delta L$ by $(\Delta t)c$ leads to

$$\varepsilon = \frac{\partial}{\Delta t\ c}$$

Since the point traveled a distance $\partial$ during a time $\Delta t$ it had a velocity

$$v = \frac{\partial}{\Delta t}$$

and we find

$$\varepsilon = v\,\frac{1}{c}$$

Thus, the strain at a point in the rod material is proportional to the particle velocity of the same point. The relation can be expanded to cover stress

$$\sigma = v\,\frac{E}{c}$$

or force

$$F = v\,\frac{EA}{c}$$

The proportionality constant $\frac{EA}{c}$ is often referred to as an impedance because it is that force with which a pile opposes a sudden change of velocity by one unit.

Example 1:

A pile is directly hit by a mass which has a velocity of 10 ft/s. What is the maximum force in the pile during impact ($E$ =30000 ksi 210kN/mm$^2$, $A$ = 15.5 in$^2$ (12HP53) 115cm$^2$, $c$ = 16,800 ft/s 5,120m/s).

Answer:

During the first instant of impact the particle velocity of the pile top is also 10 ft/s

$$\text{Thus:}\quad F = 10\,\frac{30000\ (15.5)}{16800} = 277 \text{ kips (1.26MN)}$$

## 2.2 Wave Speed

Let us now consider (Figure 2) the balance between the force

$$F = \varepsilon AE$$

acting at a cross section and the resulting acceleration

$$a = \frac{v}{\Delta t}$$

of a piece of rod of mass density $\rho$ and length $\Delta L$. According to Newton's second law

$$F = m\ a$$

which becomes (since $m = A\rho\Delta L$)

$$\varepsilon AE = \frac{v}{\Delta t}\ A\rho\Delta L$$

but $v = c\varepsilon$ and $\frac{\Delta L}{\Delta t} = c$ and therefore

$$\varepsilon E = c^2\ \varepsilon\ \rho$$

and the wave speed becomes:

$$c = \sqrt{E/\rho}$$

Example 2:

Compute the wave speeds for steel (E - 30,000 ksi, $\gamma$ = 492 lbs/ft$^3$), concrete (E = 5000, $\gamma$ = 150 lbs/ft$^3$), timber (E = 2000 ksi, $\gamma$ = 50 lbs/ft$^3$) and water (E = 295 ksi, $\gamma$ = 62.4 lbs/ft$^3$). $\gamma$ is the specific weight of the pile material.

Answer:

$$c = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{E}{\gamma}\,g} = \sqrt{\frac{E\,[\text{ksi}]\quad 144000\ (32.17)}{\gamma\ \text{lbs/ft}^3}}\ [\text{ft/s}]$$

$$c = 2152\sqrt{\frac{E\,[\text{ksi}]}{\gamma\,[\text{lbs/ft}^3]}}$$

$$C_{steel} = \sqrt{\frac{30000}{492}(2152)} = 16{,}800 \text{ ft/sec } (5120 \text{ m/s})$$

$$C_{concrete} = \sqrt{\frac{5000}{150}(2152)} = 12{,}420 \text{ ft/sec } (3800 \text{ m/s})$$

$$C_{timber} = \sqrt{\frac{2000}{50}(2152)} = 13{,}610 \text{ ft/sec } (4150 \text{ m/s})$$

$$C_{water} = \sqrt{\frac{295}{62.4}(2152)} = 4{,}680 \text{ ft sec } (1430 \text{ m/s})$$

Another relation is sometimes helpful. Using $c = \sqrt{E/\rho}$ one can express the quantity, $\frac{EA}{c}$, in terms of the basic material properties:

$$\frac{EA}{c} = \frac{EA}{\sqrt{E/\rho}} = A\sqrt{E\rho}$$

This expression clearly indicates that equal changes of E or $\rho$ have identical effects on the impedance.

3. Non-Uniform Piles

Suppose that a pile has a splice at midlength and that its properties change below this splice (Figure 3). The top and bottom sections may have impedances $I_{top}$ and $I_{bot}$, respectively. When the stress wave ($v_{impact}$) arrives at the splice a reflection ($F_{up}$, $v_{up}$) will be generated such that forces and velocities are in balance just above (subscript top) and below (subscript bot) the splice.

Because of the reflection we have

$$v_{top} = v_{impact} + v_{up} = v_{bot} \quad \text{(continuity)}$$

$$F_{top} = I_{top}\, v_{impact} - I_{top}\, v_{up} = I_{bot}\, v_{bot} \quad \text{(equilibrium)}$$

Solving simultaneously

$$v_{bot} = v_{impact} \frac{2\, I_{top}}{I_{top} + I_{bot}}$$

and therefore

$$F_{bot} = v_{impact} \frac{2\, I_{top}\, I_{bot}}{I_{top} + I_{bot}}$$

or

$$F_{bot} = F_{impact} \frac{2\, I_{bot}}{I_{top} + I_{bot}}$$

Example 3:

A nonuniform pile consists of two steel sections. The lower section has a cross sectional area which is only one half that of the top section portion. What are force and particle velocity in the lower section as a function of the impact quantities in the upper section?

Answer:

$$v_{bot} = v_{impact} \frac{2\, I_{top}}{I_{bot} + I_{top}} = \frac{2\ (1)}{1/2 + 1} = \frac{4}{3}\, v_{impact}$$

$$F_{bot} = F_{impact} \frac{2\, I_{bot}}{I_{bot} + I_{top}} = \frac{2\ (1/2)}{1/2 + 1} = \frac{2}{3}\, F_{impact}$$

As it will be shown later, the wave force in a pile is essential for overcoming soil resistance forces. Thus, a reduction in cross sectional area like in Example 3 can essentially influence the pile's driving behavior.

Another example of a "non uniformity" is a free end of a pile. This means $I_{bot} = 0$. Therefore

$$v_{bot} = 2\, v_{impact}$$

and of course

$$F_{bot} = 0 \text{ (as required for a free end)}$$

The effect of "fixing" the pile bottom can be studied in a similar fashion. Since this means that $I_{bot}$ becomes infinitely large

$$v_{bot} = 0$$

and

$$F_{bot} = 2\, F_{impact}$$

4. Extension of Reflection Theory to a General Situation

Suppose we investigate forces and velocities at a pile point, k, which is located between two sections, top and bot through which the wave travel time, $\Delta t_i = \Delta L_i / c_i$ is equal. A resistance $R_k$ may act at k. In addition, we know at time $t_1$ the forces in the waves that travel downward, $F_{d,top}$, the upper section and upwards, $F_{u,bot}$, the lower section. The two waves meet at k at time $t_2 = t_1 + \Delta t_i$. If we use again the conditions of equilibrium and continuity we find

$$\begin{aligned} F_{d,bot}(t_2) &= F_{d,top}(t_1)\,(2z_{bot}) \\ &\quad + F_{u,bot}(t_1)\,(z_{top} - z_{bot}) \\ &\quad - R_k\,(z_{bot}) \end{aligned}$$

$$\begin{aligned} F_{u,top}(t_2) &= F_{u,bot}(t_1)\,(2z_{top}) \\ &\quad + F_{d,top}\,(z_{bot} - z_{top}) \\ &\quad + R_k\,(z_{top}). \end{aligned}$$

The various terms are shown in Figure 4 at point k. The waves generate a force

$$F_k = F_{d,top}(t_1) + F_{u,top}(t_2)$$

and velocity

$$v_k = (F_{d,top}(t_1) - F_{u,top}(t_2))/I_{top}$$

Integrating this velocity and expressing $R_k$ as a function of velocity and displacement leads to a wave equation approach after Fisher. A soil model like the one proposed by Smith (see also Sections 5 and 7) may be used.

5. Soil Resistance Forces

The soil opposes the pile penetration both through friction at the shaft and through point resistance. In order to achieve a permanent pile set a shear failure has to be induced both at the pile - soil interface and at the pile bottom. In addition the soil must be displaced at the bottom.

Experience has shown that the static load bearing capacity is related to the driving resistance. However, additional considerations must be taken in the dynamic case:

(a) The static capacity may vary slowly after the driving process if finished. It may increase (freeze) or decrease (relaxation). Reasons for these changes are pore water pressure changes, soil remolding, stress redistributions in the soil and others.

(b) During driving the pile velocity changes rapidly. Thus the soil displacement around the tip creates mass forces (acceleration related forces).

(c) The driving process causes resistance forces at both the pile skin and the bottom which are substantially increased because of a high rate of shear.

The usual approach to model soil resistance forces is

(a) To predict freeze or relaxation from either soil mechanical considerations or from experience or by redriving the pile after a waiting period and observing the difference in driving behavior.

(b) The mass related soil resistance forces are neglected.

(c) The velocity dependent increase of the shear resistance forces are modeled by a linear dashpot. Actually, these dashpots are used in pile dynamics to account for all energy losses in the soil that are otherwise neglected (like the inertia forces of (b)).

The static resistance which is thought to be present both during driving and during the later static loading is modeled elasto-plastically. This approach is certainly justified in light of the uncertainties in the dynamic resistance forces.

Figure 5 shows actually measured soil resistance forces as they occurred under the toe plate during both driving and a static load test of (a) a 3 inch pile in sand and (b) for a 12 inch pile in silt and clay. The difference between the static and the dynamic curves is due to dynamic effects. For the case of sand the difference is very well accounted for by the linear dashpot. Case (b), however, presents large errors with this model.

The soil resistance force, R, which is the sum of the elasto-plastic static portion, S, and the dashpot force, D, can be written as

$$R = D + S = J_v(v) + S$$

where $J_v$ is a damping parameter [kips/ft/sec], and v is the pile velocity [ft/sec] at the point where R acts. S can be written as

$$S = \begin{cases} k_s\ (d) & \text{for } d \leq q \\ S_u & \text{for } d \geq q \end{cases}$$

as long as unloading (velocity $v<0$) does not occur. $S_u$ is the ultimate static resistance at a point, $k_s$ is the soil stiffness, d the pile displacement, q is the so-called quake, i.e. that pile displacement at which the soil becomes plastic.

Damping parameters are often used in a slightly different form. While $J_v$ is a viscous damping factor, the usual approach in pile dynamics is to use

$$D = J(v)S$$

with J being the so-called Smith damping factor (thus, $J = \frac{J_v}{S}$ and has dimensions sec/ft). Another way to formulate a damping constant is by dividing it by the pile's impedance. Then

$$D = J_c \frac{EA}{c} v$$

Note that $J_c$ is truly dimensionless. It is referred to as the Case damping constant.

An important advantage of Case damping is that it allows a damping force to be modeled where no static resistance force exists as in soft soils. Also for analyses of changing static resistance values the damping parameters are kept constant.

## 6. A Pile With Resistance at the Point Only

Suppose, an impact wave with velocity $v_{impact}$ and force $F_{impact} = I v_{impact}$ reaches a free pile end. It was previously found that a reflection wave would be generated such that $v_{toe} = 2\ v_{impact}$. If the pile end moved through soil it would activate a resistance force

$$R = J_v\ v + S \text{ or } R = (J_v + 1)\ S$$

Since R is suddenly applied it causes an additional (but compression) wave to travel upwards through the pile having particle velocity (upwards):

$$v_R = IR$$

Thus, the resulting pile toe velocity is

$$v_{toe} = 2\ v_{impact} - IR$$

Substituting for R one obtains

$$v_{toe} = 2\ v_{impact} - I\ (J_v\ v_{toe} + S)$$

in the case of viscous damping. Solving for $v_{toe}$ one obtains

$$v_{toe} = \frac{2\ v_{impact} - S/I}{1 + J_{v/I}}$$

(For Smith damping: $v_{toe} = \frac{2\ v_{impact} - S/I}{1 + J\ S/I}$ , for Case damping;

$v_{toe} = \frac{2\ v_{impact} - S/I}{1 + J_c}$ ).

Thus $v_{toe}$ becomes zero when

$$S = 2\ I\ v_{impact} = 2\ F_{impact}$$

In other words the largest static resistance force that the impact wave can overcome is twice the impact force.

This formula does not tell how much a pile would move given $v_{impact}$, I and S (this is dependent on the hammer mass), however, it clearly gives an upper limit for S and emphasizes the importance of the impedance I. The higher the pile impedance the larger the resistance forces that can be overcome.

Example 4:

A cast in place pile (1.5 m diam., i.e. $A = 1.77m^2$, $E = 40kN/mm^2$, $C = 4080m/s$) is struck by a mass such that a 1.5m/s pile particle results. What maximum resistance can be overcome?

Answer:

$$S_{max} = 2\ (1.5)\ \frac{(40)\ (1.77)}{4080}\ 10^6 = 52{,}000kN\ (4{,}700\ tons)$$

In reality, however, the maximum resistance activated by the wave will not be much greater than 1.2 times (31,200kN) the impact force.

Another point may be of interest. We have seen that the Case damping approach leads to a bottom velocity

$$v_{toe} = \frac{2\ v_{impact} - S\ \frac{c}{EA}}{1 + J_c}$$

Substituting S = 0 (no static resistance) and $J_c = 1$ one obtains

$$v_{toe} = v_{impact}$$

In effect this means that no reflection wave is generated if a damper with constant $J_c$ = 1 or $J_v$ = I is used. In other words, the behavior of an infinitely long, uniform pile is like that of a dashpot having a damping constant equal to the pile impedance (of course, a linear dashpot is by definition a device whose resistance force is proportional to the velocity).

## 7. Effects of the Hammer to Pile Mass Ratio

The discussion of piling behavior should not be ended without touching on the subject of hammer mass effects. So far indications were that only the impact velocity was of importance in driving a pile, however, nothing was said about how much penetration was achieved by a hammer blow.

St. Venant, the father of the Wave Equation, had studied these effects and had come up with a plot (Figure 6) showing the pile top force behavior for various hammer to pile mass ratios. These plots were made for a fixed end condition. They were extended to also show the pile bottom force. Of course, as shown above, initially the pile bottom force becomes twice the impact force. L/c is the time that it takes a stress wave to travel along a rod of length and is therefore that time at which the impact is felt at the bottom.

Two important results should be observed

(a) The forces decay faster for smaller hammer masses.

(b) Upon wave reflection the pile forces increase to values higher than the impact force if the hammer mass was sufficiently large.

The penetration of a pile depends on the difference between the toe force (fixed case) and twice the static resistance force. The larger the remaining area the greater the penetration under a blow. Figure 6 shows that heavy rams can overcome a resistance force after time 3L/c even if the impact force was less than $S_{max}$. Such a situation does not often occur in reality.

## 8. The Lumped Mass Wave Equation Model

### 8.1 Transition from Continuous to Discrete Systems

In deriving the expression $c = \sqrt{E/\rho}$, Newton's Second Law was used which states that mass times acceleration equals force. We have considered very small masses $\Delta L \rho A$ and very short time increments, $\Delta t$, for this dynamic balance.

It is possible to use finite masses and finite time increments together with Newton's Second Law to compute the pile forces and displacements at a discrete number of points. Similarly, a ram can be divided into several masses. Correspondingly, the elasticity of the pile can be modeled using a finite number of springs. The errors involved are small if we consider phenomena that change slowly with respect to the wave travel time in one element.

From measurements it was found that the forces and velocities in the pile usually rise to their peak value in more than one millisecond. For such rise times a segment length of five feet is sufficiently short. On the other hand, diesel hammer rams which impact directly against a steel anvil exhibit shorter rise times and a shorter segment length is recommended.

### 8.2 The Pile/Soil Model

The pile is modeled by a series of masses which are connected by springs and dashpots. Each mass is also supported (or its motion is resisted) by an elasto-plastic spring and a dashpot which together represents the soil action (Figure 7).

Spring constants, k, for the pile (and similarly for hammer components) are computed using the relation

$$k = \frac{EA}{\Delta L}$$

where $\Delta L$ is the length of a segment (or thickness of a cushion).

The masses are derived from

$$m = A \rho \Delta L$$

Two important relations should be discussed

$$\sqrt{km} = \sqrt{\frac{EA}{\Delta L} A \rho \Delta L} = A\sqrt{E\rho} = \frac{EA}{c}$$

$$\sqrt{\frac{m}{k}} = \sqrt{\frac{A \rho \Delta L}{EA} \Delta L} = \frac{\Delta L}{\sqrt{E/\rho}} = \frac{\Delta L}{c}$$

Thus, the impedance of a pile segment can be determined from the square root of the product of mass and stiffness. This fact makes it easy to compute the viscous damping factor from Case damping constant given mass and stiffness of a segment.

Also important is that the ratio $\Delta L/c$ (which is the time that it takes the stress wave to travel through the segment) can be determined from mass and stiffness values. $\Delta L/c$ is also called the critical time, $\Delta t_c$. Note that the computational time increment $\Delta t$ has to be smaller than $\Delta t_c$, a fact that is considered in most programs.

Spring and mass values are easily obtained from the equations stated. The accuracy of the values is essential for a meaningful answer. The dashpots within the pile model are of lesser importance. Their constants are not well known and are, as far as WEAP is concerned, based on the pile's impedance. Calling $c_p$ the dashpot constant between two pile elements.

$$c_p = c_{pp} \frac{EA}{c} \frac{2}{100}$$

where $c_{pp}$ is the percentage of critical damping in percent (critical damping is here defined as $2\ EA/c = 2\sqrt{k\ m}$ for the spring next to and mass below the dashpot). It is recommended to use values not larger than between one and three (3) percent.

9. SUMMARY

The discussion of the hammer induced stress wave has yielded a number of important relations which are now summarized.

1. $F = v \frac{EA}{c}$ — Force and velocity in a stress wave are proportional. The proportionality constant is equal to the impedance EA/c.

2. $c = \sqrt{E/p}$ — The wave speed increases when Young's modulus increases and decreases when the material density increases.

3. $v_{bot} = v_{impact} \frac{2 I_{top}}{I_{bot} + I_{top}}$ — The velocity of the impact wave increases when the pile impedance decreases.

4. $F_{bot} = F_{impact} \frac{2 I_{bot}}{I_{bot} + I_{top}}$ — The force of the impact wave decreases when the pile impedance decreases.

5. $v_{toe} = \frac{2 v_{top} - S \frac{c}{EA}}{1 + J_c}$ — The velocity of the pile bottom can be expressed in closed form if no skin friction is present.

6. $S_{max} = 2 \frac{EA}{c} v_{impact}$ — The maximum resistance force that an impact wave can overcome is twice the impact velocity times the pile impedance.

7. $J = \frac{J_v}{S}$ — Smith's damping is equal to the viscous damping constant divided by the static soil resistance.

8. $J_c = \frac{J_v}{EA/c}$

Case damping is equal to the viscous damping constant divided by the pile's impedance.

9. $k = \frac{EA}{\Delta L}$

A spring stiffness is calculated as Young's Modulus times cross sectional area divided by the segment thickness (or length).

10. $m = A \rho \Delta L$

An element mass is the product of cross sectional area, mass density and element thickness (or length).

11. $EA/c = \sqrt{k m}$

The pile impedance in the discrete case is equal to the square root of the product of segment stiffness and mass.

12. $\Delta t_c = \sqrt{m/k}$

The critical time increment for a lumped mass analysis is given by the square root of the ratio of element mass to stiffness.

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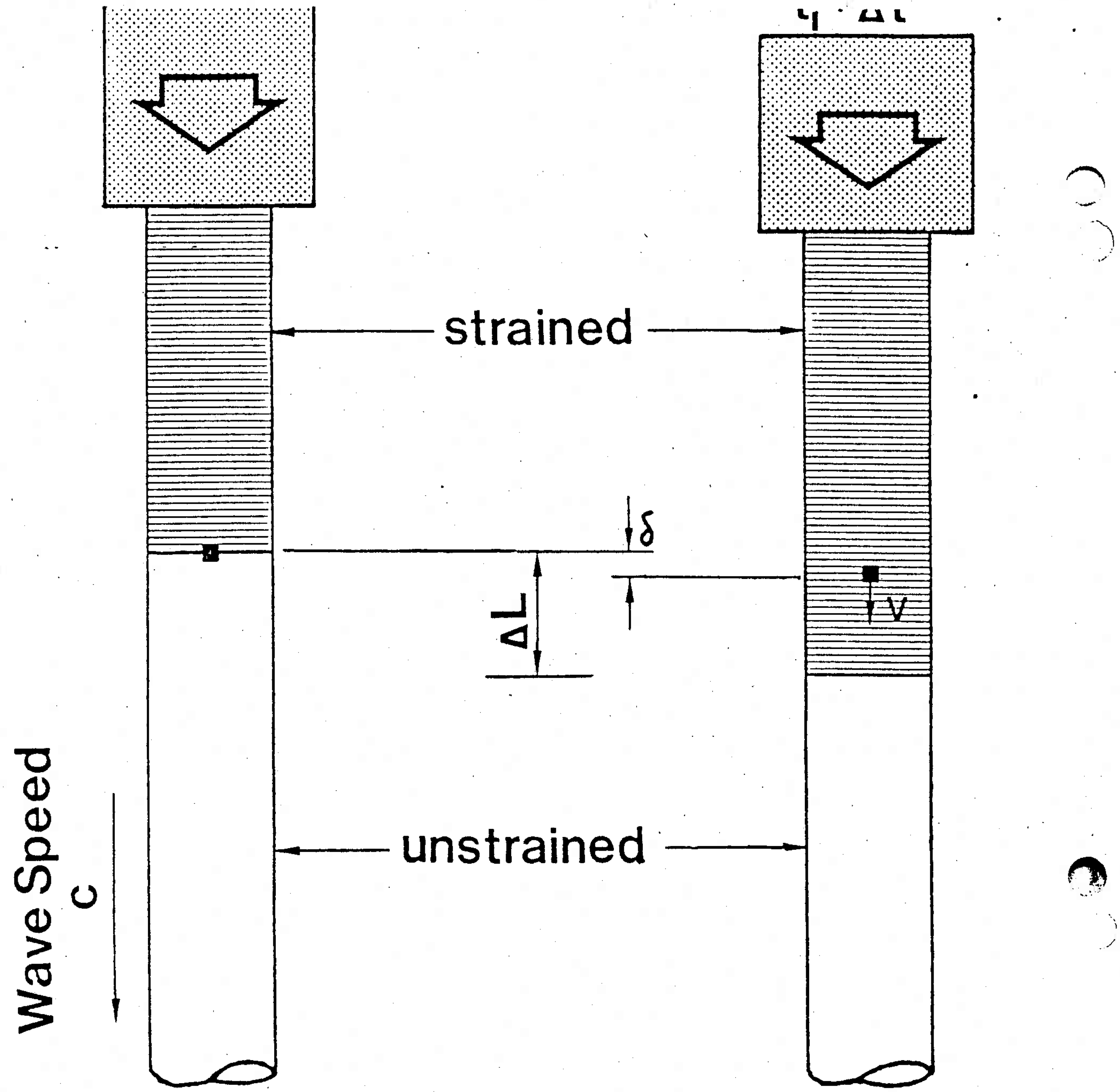


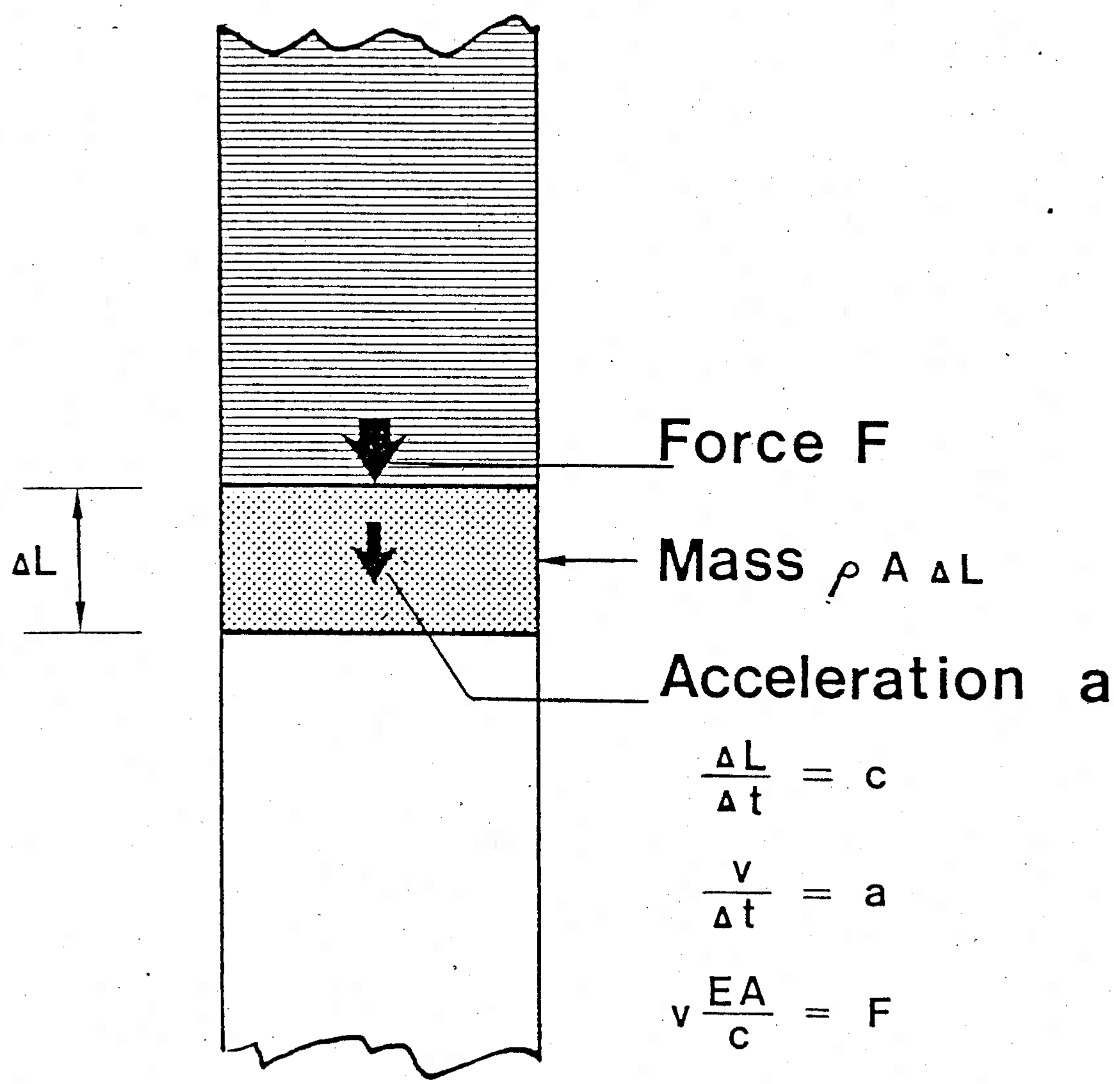

$$\Delta L = c\Delta t$$

$$\varepsilon = \frac{\delta}{\Delta L} = \frac{\delta}{c\,\Delta t}$$

$$\frac{\delta}{\Delta t} = v \qquad \text{(particle velocity)}$$

$$\therefore \quad \varepsilon = \frac{v}{c}$$

Figure 1: Derivation of Velocity - Force Proportionality




$$\frac{\Delta L}{\Delta t} = c$$

$$\frac{v}{\Delta t} = a$$

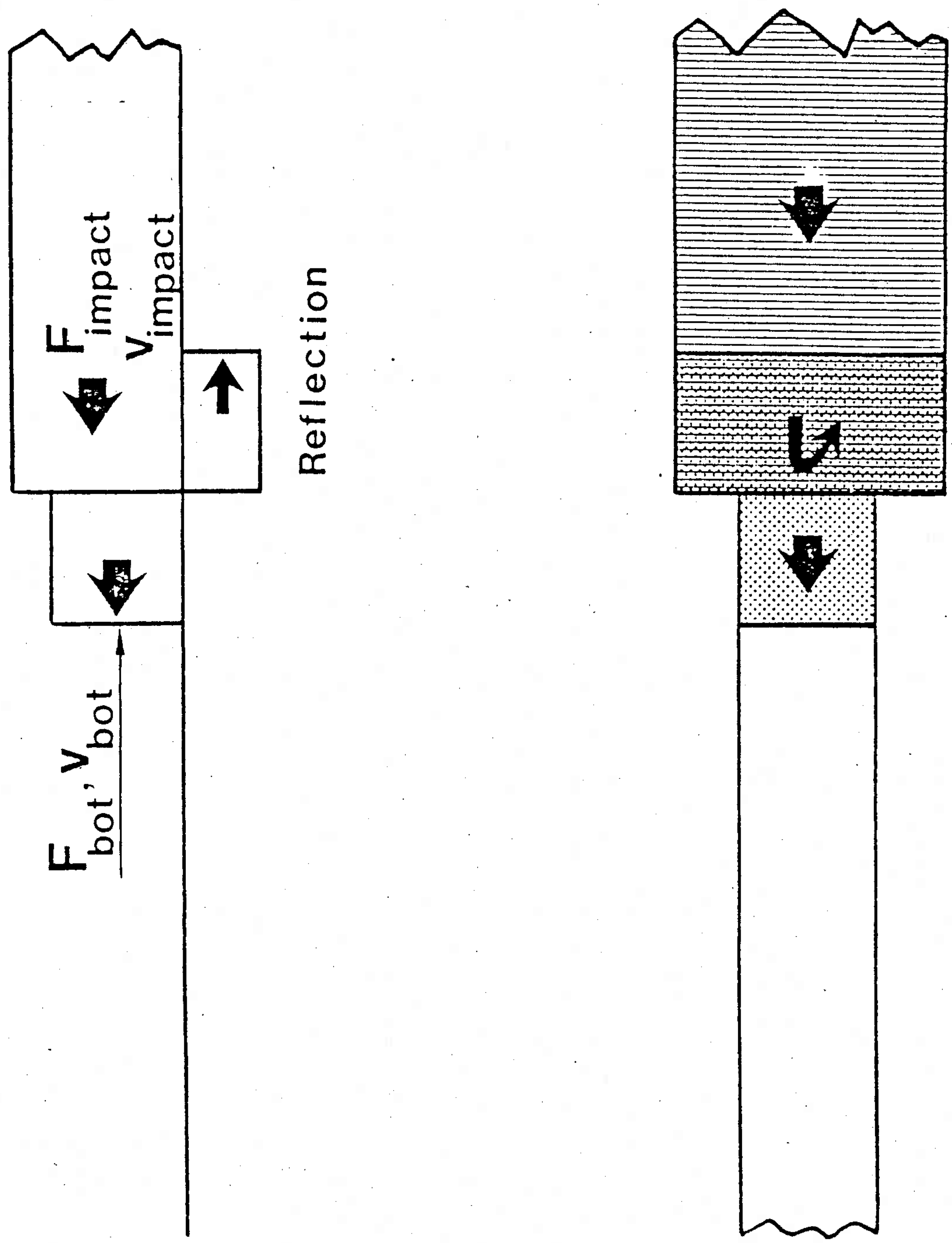
$$v\,\frac{EA}{c} = F$$

$$\text{Force} = (\text{Mass})(\text{Acceleration})$$

$$v\,\frac{EA}{c} = \rho A \Delta L\,\frac{v}{\Delta t}$$

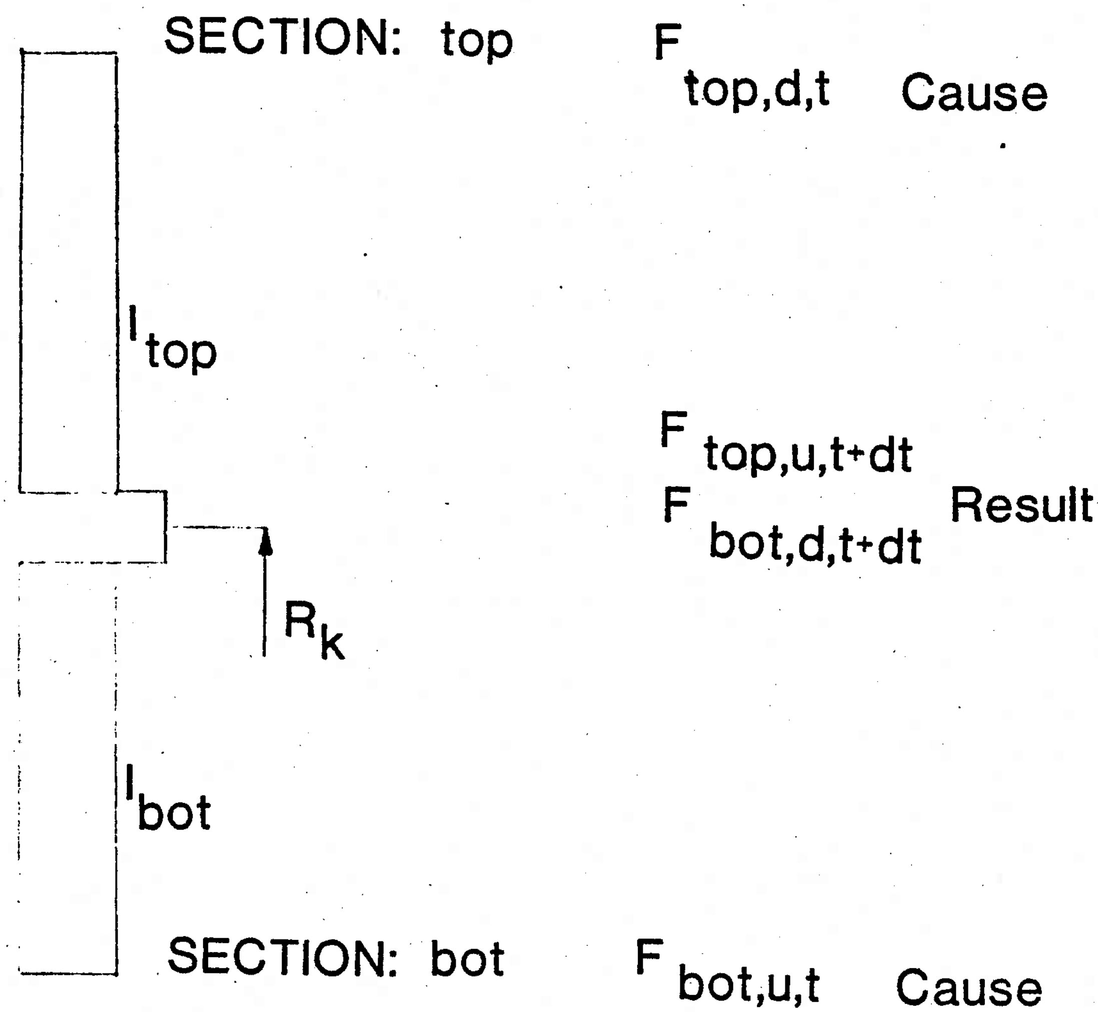
$$\boxed{c^2 = \frac{E}{\rho}}$$

Figure 2: Derivation of Wave Speed c



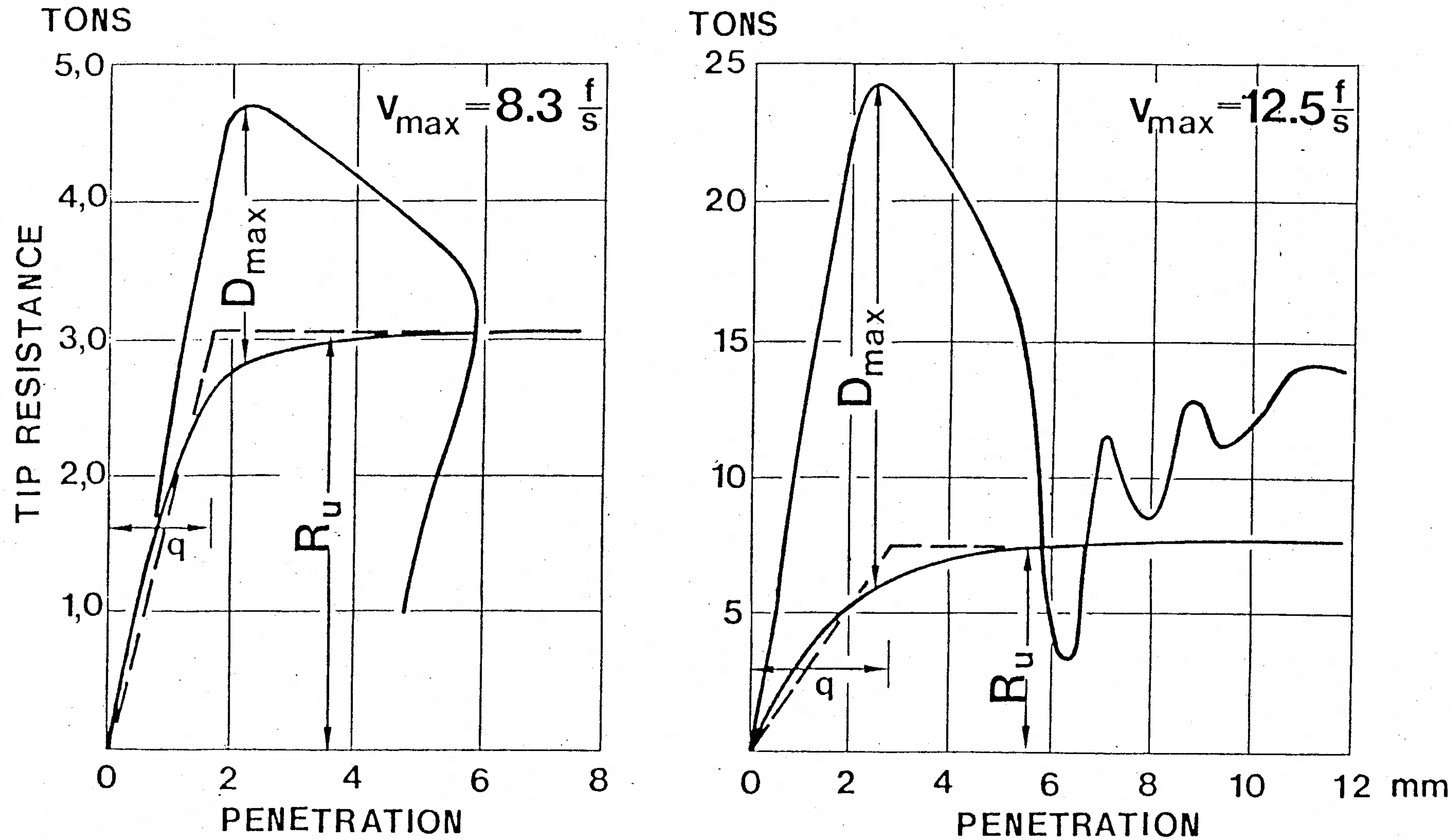

WAVES IN A NONUNIFORM PILE

Figure 3: Derivation of Wave force and particle velocity below a change in cross section




$$Z_{BOT} = \frac{I_{TOP}}{I_{TOP} + I_{BOT}} \qquad Z_{TOP} = \frac{I_{BOT}}{I_{TOP} + I_{BOT}}$$

Figure 4: Wave Interaction at two short, uniform pile elements




Fine Sand
3 inch dia. pipe

Clayey Silt
12 inch dia. pipe

Figure 5: Measured Soil Resistance - Static and Dynam

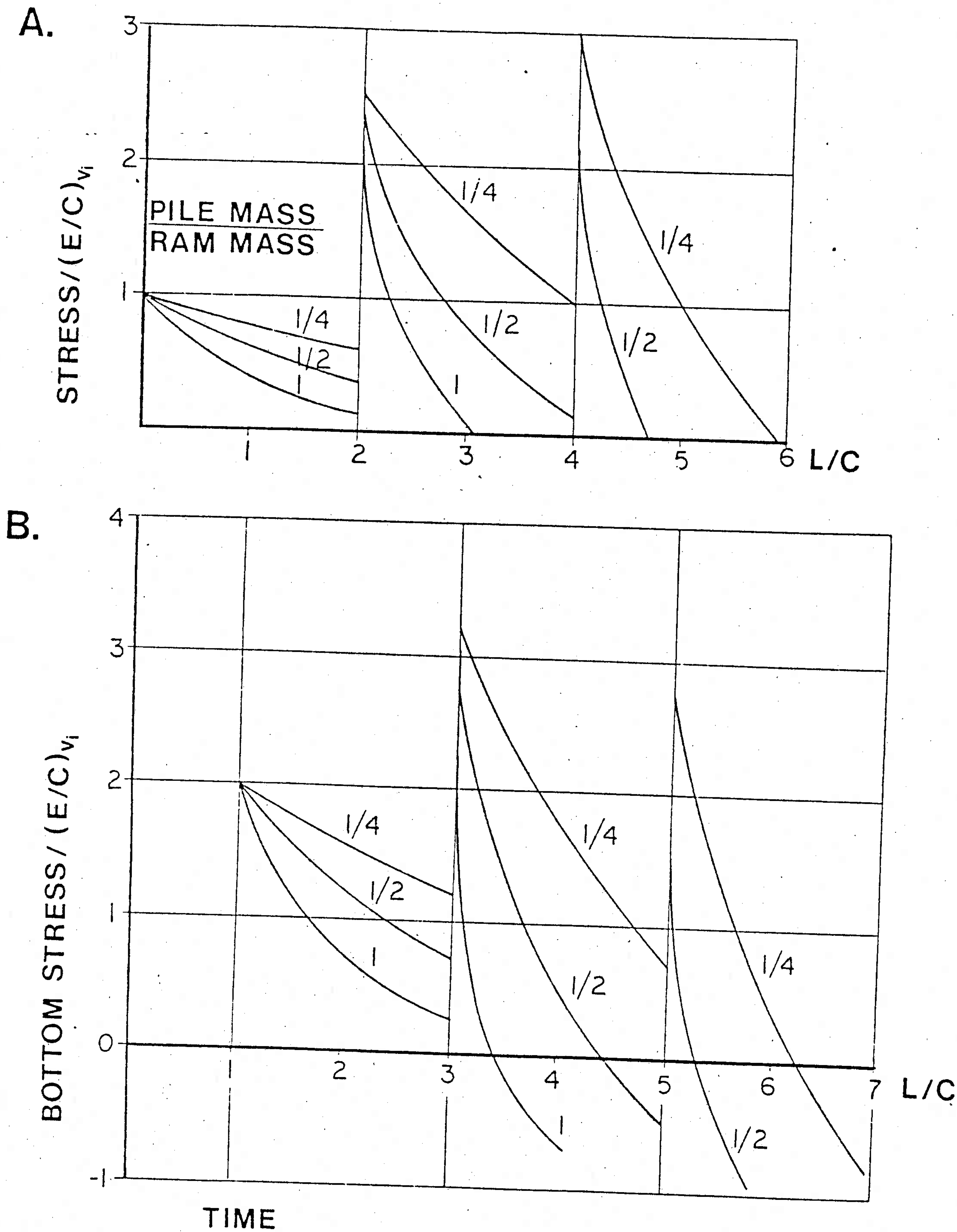



Figure 6: St. Venant Solution

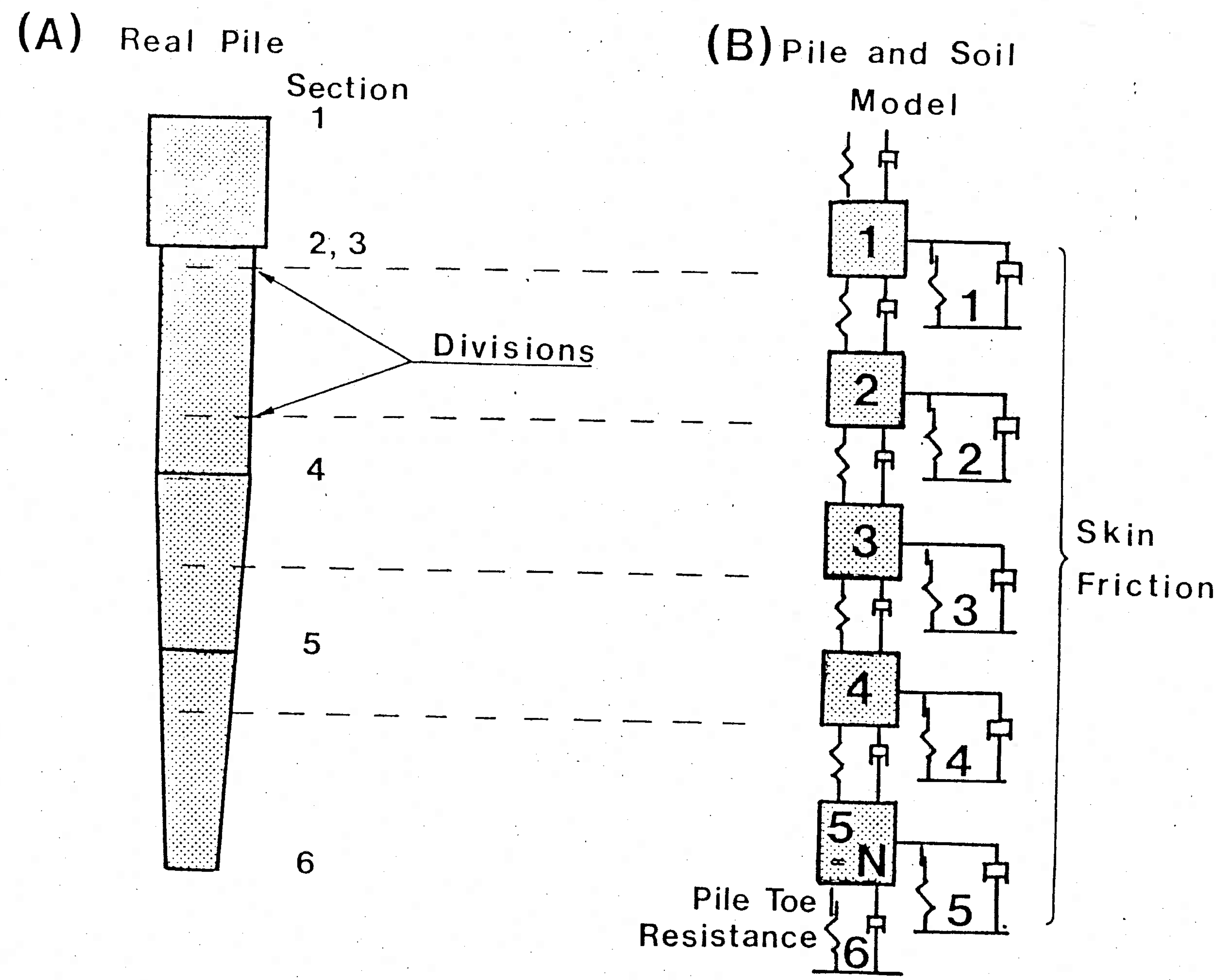



Figure 7: Pil[illegible]l Model