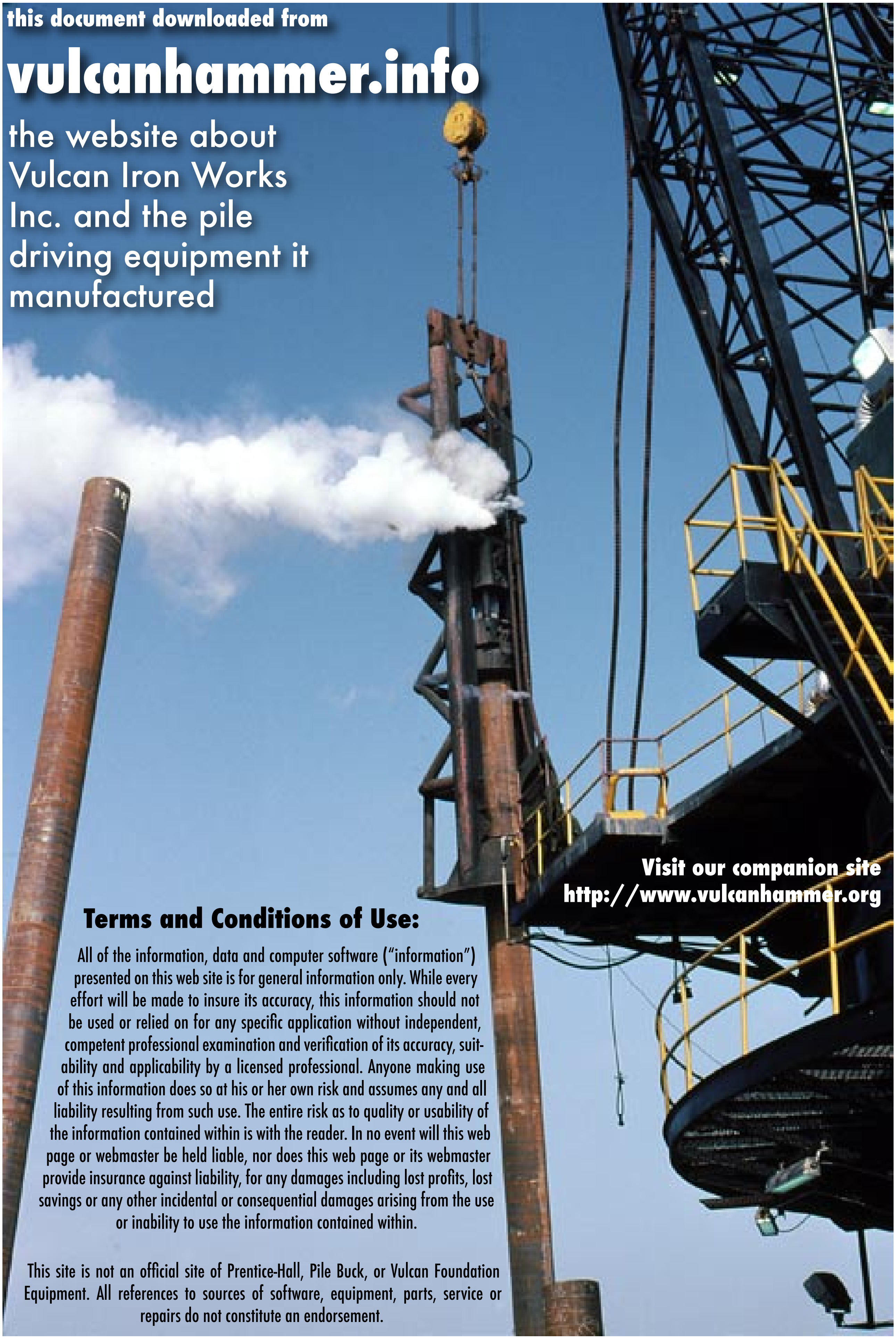

Persistent Issues in the Wave Equation for Piling for Both Forward and Inverse Methods[1]

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Abstract

This article is an overview of the current state of both forward and inverse analysis of wave propagation in piling. It begins with a summary of the typical acceptance procedure for the wave equation as applied to (primarily) driven piles. It then defines and describes what are forward and inverse methods, outlining criteria which are important for success. After this the governing equations are discussed, both undamped and damped (Telegrapher's) wave equations, and why it is important to consider the latter as the true governing equation for pile dynamics. This is followed by a discussion of explicit and implicit methods and how they are (and might be) applied to the problem at hand. The difference between finite difference and finite element methods is discussed, and how each has been applied in either a one-dimensional or two-dimensional way. Finally the issue of rheology is examined. The central problem with dynamic analysis–the inability to separate static and dynamic resistance by the basic inverse methods available–is discussed in detail.

1. Introduction

The one-dimensional equation has been applied to driven piles for more than eighty years now. The key event in the process was the development of a finite-difference numerical method (Smith (1960)) to analyse the wave propagation. Once this was done, it was possible to realistically model pile behaviour during driving, and to obtain some idea of the pile resistance during driving, and by extension its capacity after driving is complete.

[1]Originally posted at vulcanhammer.info May 2013. Document subject to terms and conditions of the website.

The pattern of development and implementation of these methods is fairly predictable and follows a pattern such as this:

1. Development of the technique, be it a forward or inverse technique (this will be explained shortly.) The development includes the computer software and hardware necessary to carry out the actual "number crunching" which comes with numerical techniques.

2. Application of the method to actual projects.

3. Comparison of the results of the method to the actual project data. The usual thing we see in the literature is comparison with static load tests.

4. Conclusion that either a) correlation has been achieved or b) there are good reasons why we do not have correlation. The most satisfactory result is obviously (a). Part of this process is in the elimination of "outliers" from the data.

5. A general effort to have the method accepted by the profession, including owners, engineers, etc.

It is tempting to say that the science of analysing this wave propagation, after all of the resources expended upon its development and the wide acceptance of these results, is mature and not subject to improvement. But this is certainly not the case. The purpose of this article is to examine the issues at hand from a different angle, i.e., the theoretical consistency of the methods being currently used. In doing this it is hoped that some light might be shed on the improvement of these methods and the further enhancement of the design and verification of deep foundations.

This article will discuss topics which are probably not in the ongoing familiarity of geotechnical engineers, even those which are primarily engaged in the design, construction and verification of deep foundations. Thus, the level of discussion may not be as in-depth as specialists in numerical methods would like. Nevertheless, it is hoped that this will give more of those involved a better feel for the issues at hand, which should help to direct the discussion towards more productive solutions.

2. Numerical Methods and Geotechnical Engineers

Wave propagation in driven piles, its prediction and analysis, is a field that is very specialized, even within geotechnical engineering in general and the design and construction of deep foundations in particular. Many very competent geotechnical engineers are unfamiliar with the basic theory behind the wave equation as applied to piles, let alone the actualization of that theory. This is not to disparage the profession; most geotechnical engineering involves a different skill set from structural dynamics and numerical methods that would–and do–trip up the latter relative to the former.

Some of the gap has been bridged in recent years by two events. The first is the more general requirement of geotechnical engineers (this of course dependent upon their location) to be familiar with earthquakes and their effect. Earthquake movement in soil is also a wave propagation problem, one that has some similarities with the wave equation in piles but also some important differences. In many cases codes and methodologies have been developed that shield the practising geotechnical engineer from the direct application of wave dynamics. Additionally the growing employment of geophysical methods, both in large- and small-scale applications, gives practitioners another exposure to wave propagation analysis.

The second–which is empowering and shielding at the same time–is the use and acceptance of numerical methods in the practice of civil engineering. On the whole, the implementation of these, like the adoption of many other new methodologies in civil engineering (LRFD comes to mind) has lagged the rest of the profession. This is not only because of the variable nature of soils, rocks and IGM's as engineering materials, but also because of the highly non-linear nature of the problems in soil mechanics and foundations.

The implementation of these, however, has brought a new problem to the forefront: the fact that many practitioners, through no fault of their own, are unfamiliar with the difficulties that come with the use of numerical methods. These include machine computation errors, discretization difficulties, ill-conditioned problems, convergence and stability issues, and the like. Most commercial codes are designed to shield the user from these problems, but in the process some of these techniques introduce errors of their own. If more practitioners were familiar with the possible difficulties in machine computation and problem discretization, they would be more critical in the acceptance of their results.

Given all of this, the long history of the wave equation in piles is remarkable in its early development as a numerical method, and even more remarkable in the acceptance of the technique for common practice. This acceptance, however, should not blind us to the difficulties inherent in any numerical method, especially one that models a non-linear system such as exist in high-strain pile dynamics.

3. Forward and Inverse Methods

Now that we have "set the stage," we can proceed and discuss some definitions and basic concepts. The first question is suggested by the title: what are forward and inverse methods?

The simplest way to explain this is by using an example. Let us consider a simple frame structure to which loads are applied. Whether we solve this problem by "classical" methods (energy methods, etc.) or a method such as finite element analysis, we construct a model of the structure and apply the loads to determine the deflections and stresses (the latter via the moments and axial forces) of the structure. Such a method is a forward method; we model the structure and apply the loads, and from this we obtain the result.

Now let us consider the case of an existing structure where we would like to determine the actual loads on the structure. Again we model the structure, but then apply the results (stress, deflection, etc.) and from that determine the loads. Such an analysis would employ an inverse method: given the results, we analyse the structure to obtain the input data. In reality such an analysis would be useful if, for example, the structure was showing distress and we wanted to determine the magnitude, direction and nature of the loads that might be causing this distress.

Broadly speaking, in design we use forward methods, and in verification we use inverse methods. Neither of these methods has to be numerical; in fact, a great deal of forensic engineering is done without recourse to numerical methods, but which is, in reality, an inverse type of analysis.

Turning to pile dynamics, the wave equation is the forward method we use to analyse the driving of piles. We model the hammer, pile and soil system, and then perform the analysis. Even here, however, we see the beginnings of inverse methodology. The "bearing graph" is a method by which we analyse a variety of pile resistance profiles, the object of which is to relate the performance of the system to a variety of possible results, in this case pile resistances. The uncertainties of both the ground itself and the static capacity methods we have at our disposal make a rapid transition to inverse analysis a necessity. In the early years of the numerical wave equation, and even before that, this type of "inverse" analysis was employed, and we still see some of this type of application of the wave equation program today (Rausche, Nagy and Webster (2009).)

Inverse methodology even pre-dates the wave equation in pile dynamics. With the dynamic formulae, it was possible to simply rearrange the equations and/or develop a bearing graph type of result (something hammer manufacturers, for example, did routinely for their products) to determine the blow count from the capacity (forward) or the capacity from the blow count (inverse). Obviously the deficiencies in the dynamic formulae made these results problematic, but the need for a reliable inverse methodology in pile dynamics is inherent in the application.

Since we are using a numerical method for the forward analysis of the wave equation as applied to piles, it makes sense that the inverse method be likewise numerical. Obviously we would like our methods, forward and inverse, to accurately model the physical system at hand and to give results that inform us of the characteristics of the system. To achieve these results, there are two things we would like to see in both forward and inverse methods.

The first is that the result is **unique**. With the forward method, uniqueness can be guaranteed fairly readily, if we consider the effects of modelling error and misrepresentation of the physical system. With the inverse method, things are not so clear. This has been an issue since Rausche et.al. (1972), and specifically the response by Screwvala (1973). In reality Screwvala's objections were broader than the issue of uniqueness, and the uniqueness of CAPWAP results have been challenged since Screwvala (1973) (Danzinger et.al. (1996).)

Formal uniqueness is difficult to achieve, especially with a time-dependent, non-linear system such as we have with driven piles. Another approach is either to a) develop solutions where the result is the most likely outcome or b) develop a range of results from which we can say that one or more of them are valid, from which we apply other criteria to come to a satisfactory result. In doing this, one must be careful to avoid a subjective method of solution search (Balthaus (1988).)

The second is that the inverse method, given the results of the forward method, should **produce the same result**. For example, if we were to run the wave equation program for a given hammer/pile/soil system and then take the pile top results (acceleration/velocity/strain/displacement) and put them back into the inverse method, the latter should return the same result (in this case the soil resistance distribution and static/dynamic resistance division) as the forward method. This is a basic requirement. However, in the urge to correlate the predictive results with the results of, say, static load tests, this kind of analysis rarely appears in the literature except when mediated by actual pile driving results. With these it is possible to upgrade the forward analysis and improve the situation all around, but a rigorous comparison of forward and predictive results by themselves is not common.

One way to help make this outcome a reality is for the forward and inverse methods to be "mirror images" of each other. Because of the difficulties inherent in the problem, this has generally not been considered in this application. For example, it was common–especially in the early years of modern pile dynamics–to concentrate the pile resistance at the toe. Obviously the resistance along the shaft responds differently to the stress-wave than that at the toe. A more symmetrical analysis system would go a long way to solving the other issues in dynamic analysis of piles.

4. The Governing Equation

No physical system can be modelled without a proper formulation of the governing equation. For the undamped one-dimensional wave equation, that is as follows:

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2\frac{\partial^2}{\partial x^2}u(x,t) \tag{1}$$

where $u(x,t)$ is the displacement along the pile as a function of distance x from the top (customarily zero) and time t (impact time zero) and c is the acoustic speed of the pile. The "classic" solution of this is d'Alembert's equation (Sobolev (1964))

$$u(x,t) = \psi_1(x - ct) + \psi_2(x + ct) \tag{2}$$

d'Alembert type solutions have been applied to piles since Isaacs (1931) and Glanville et.al. (1938). Both numerical and closed-form solutions such as Warrington (1997), and of course semi-infinite pile theory upon which the whole impedance concept is based, are derived from this equation. They have been especially popular with graphical solutions such as that of Fischer (1960). They appear routinely in the literature for the Case Method and CAPWAP (Rausche (1970); Rausche et.al. (1972); Rausche, Goble and Likins (1985).) They also appear in alternatives such as Liang (2003). The advent of numerical methods has largely superceded the unaided use of d'Alembert type solutions; the "bookkeeping" necessary to keep up with the upward and downward travelling waves was considerable.

Nevertheless the problem with (2) goes deeper than d'Alembert solutions. The more fundamental problem is that the equation above assumes no dampening or resistance of any kind along the shaft of the pile. Although many driven piles have exposed portions which do not contact the soil, it is the rare driven pile (or any other type of deep foundation) which lacks shaft resistance of any kind.

A more accurate description of the wave equation for piles would be the Telegrapher's wave equation (Webster and Pimpton (1966),) given by

$$c^2 \frac{\partial^2}{\partial x^2} u(x,t) = \frac{\partial^2}{\partial t^2} u(x,t) + a u(x,t) + 2\, b \frac{\partial}{\partial t} u(x,t) \tag{3}$$

where a, b are the distributed spring and dampening constants along the pile shaft. Obviously, if a, $b = 0$, then this equation reduces to the undamped form. So what we have is not a contradiction as much as an expansion, albeit an important one.

There are limitations to (3) as well. First, it assumes a uniform cross-section of the pile, as well as no discontinuities in the pile. Both of these assumptions are present in the undamped solution as well. Neither of these is necessarily true of any driven pile, and discovery of the latter is the main driving force behind the application of wave propagation theory to pile integrity testing.

Beyond these limitations, the Telegrapher's equation assumes a linear soil response for both elasticity and dampening along the pile shaft. Neither of these (especially the former) can be counted on with driven piles, and in reality the whole point of pile driving is to push the soil beyond its elastic limit and allow the pile to achieve a permanent set with each blow. This is the inherent weakness of such approaches as Pao and Yu (2011). As was the case with the application of Winkler theory to lateral pile loading and response, the soils simply do not respond to their mobilization in linear ways.

Nevertheless, in spite of these limitations, the Telegrapher's equation is closer to the realities of driven piles than the undamped equation. This is significant in the development of new methods for the dynamic analysis of driven piles.

5. Numerical Methods for Pile Dynamics

5.1. Overview

It is a credit to the deep foundations community that the numerical analysis of wave mechanics in piling during impact installation was one of the first, if not the first, civil engineering application to be quantified using a digital computer applied to numerical analysis. In retrospect, however, it is amazing that the quality of the results were as high as they were given the methodology and the way it was applied. To understand this we need to delve a little into numerical integration.

When most people in the deep foundations business refer to the "wave equation," they are referring to a computer program of one kind or another. In the forward method what that program does is solve a non-linear version of (3) with appropriate modelling of the hammer, cushion, driving accessory and pile toe response. The pile itself is divided up (or discretised, to use the fancy term) into finite segments, which also enables one to vary the soil properties along the shaft in a straightforward way, i.e., to assume that they are constant over the segment but perhaps different from one segment to the next. In the inverse, the force-time and acceleration-time or displacement-time history at the pile top is known; the soil properties and resistance distribution is determined based upon the response of the pile to the impulse at the pile top.

Because we are dealing with a dynamic (time-varying) phenomenon, we also have to discretize the time as well as the distance. This is done through what are referred to as "time-marching" schemes (Lomax et. al. (2003)) of one kind or another. For one-dimensional analyses of pile dynamics, the whole process is relatively simple compared to two- or three-dimensional problems in such fields as fluid dynamics or solid mechanics. The time step we choose depends upon both the nature of the system and the numerical integration scheme we have chosen.

5.2. Smith's Numerical Method

Since it occupies a prominent place in the development of the numerical analysis of the forward method in pile dynamics, some consideration should be given to the method developed by Smith (1960). This is especially important since his technique was retained by the finite difference codes that followed it, even with their additions and improvments (Hirsch et.al. (1976); Goble and Rausche (1976, 1986).)

It is interesting to note that Smith (1955) presented his numerical method several years before we generally consider the beginning of the practical use of the wave equation for piles. The method proposed here can be fairly characterised as a backward-difference explicit Euler method of numerical integration (Rausche (1970).) One of the commenters on this paper was W.E. Milne, who suggested one of the predictor-corrector methods for which he is well known.

Although he did not go that far, Smith (1960) allowed his method to drift towards a central-difference technique (and a non-self-starting one at that) by the summation of forces for each mass, which were lumped at the bottom of each pile segment. In doing this he probably avoided some of the pitfalls of the explicit Euler method, for which, as pithily noted by Carnahan, Luther and Wilkes (1969), "accuracy limitations preclude its (explicit Euler's) use for most practical problems."

Although the limitations of this numerical integration scheme were recognised early (in addition to Milne's comments, see also Fischer (1960)) Smith's basic scheme has endured for many years. So is improvement possible? There are many avenues we can take and integration schemes we can consider, but probably the most important division of numerical integration methods for partial differential equations such as this one is between explicit (such as Smith (1960)) and implicit schemes.

5.3. Explicit and Implicit Schemes

We have referred to the scheme presented in Smith (1955) as an "explicit" scheme. So what does this mean? To attempt to understand this, we will consider the "one-way" (semi-infinite) undamped wave equation, given by the expression (Warrington (1997))

$$\frac{\partial}{\partial t}u(x,t) = -c\frac{\partial}{\partial x}u(x,t) \tag{4}$$

Use of this equation allows us to show the numerical schemes without the complications of either the second derivatives or dampening/elastic terms.

When the Explicit Euler scheme is applied to this differential equation, the result is

$$\frac{u(x,t)_i^{n+1} - u(x,t)_i^n}{\Delta t} = c\frac{u(x,t)_i^n - u(x,t)_{i-1}^n}{2\Delta x} \tag{5}$$

In this case n, $n+1$ are not powers but represent the point in time where we are "marching," n being the current time step and $n+1$ being the next one. The subscript i is the data point; the point $i+1$ is the data point "in front of" the one under consideration and $i-1$ is the one behind it.[2]

Knowing the conditions of the current point in time n, (5) can be explicitly solved for the value of $u(x,t)$ for the next time step, thus the designation explicit:

$$u(x,t)_i^{n+1} = \frac{c\Delta t}{2\Delta x}\left(u(x,t)_i^n - u(x,t)_{i-1}^n\right) + u(x,t)_i^n \tag{6}$$

There are also "implicit" schemes as well. Starting again with (4,) the Implicit Euler scheme can be written as follows:

$$\frac{u(x,t)_i^{n+1} - u(x,t)_i^n}{\Delta t} = c\frac{u(x,t)_{i+1}^{n+1} - u(x,t)_{i-1}^{n+1}}{2\Delta x} \tag{7}$$

Note that the desired quantity $u(x,t)_i^{n+1}$ cannot be solved for from terms in time n, thus the designation implicit. In simple terms, explicit schemes predict the future by computing the next step from present data, and implicit schemes compute the next step from future data.

At this point some observations are in order:

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[2]Schemes such as this can be (and usually are) derived for ordinary or partial differential equations using Taylor series expansions, which would include consideration of residual terms, but such are beyond the scope of this article.

1. Those who are familiar with Smith (1955) will recall that time steps based on the single-degree of freedom natural frequency of the stiffest point were set to avoid the destabilizing effects of an excessively long time step. This has its counterpart in (6) as limiting the value of $\frac{c\Delta t}{2\Delta x}$. Unfortunately, for this scheme applied to (4) via (6) it can be shown that (5) is unconditionally unstable at any value of $\frac{c\Delta t}{2\Delta x}$. This may explain the changes that took place as documented in Smith (1960).

2. Although there are explicit schemes that have better performance than Explicit Euler, Lomax et. al. (2003) note that none of the explicit schemes are "A-stable," which means that "it is unconditionally stable for all ODE's (ordinary differential equations) that are stable." This not only includes schemes such as explicit Euler but also predictor-corrector methods such as those suggested by Milne and the well-known Runge-Kutta techniques, one of which was employed by Bossard and Corté (1983).

3. For (7), which admittedly is not based on the "full" wave equation, the Implicit Euler scheme is unconditionally stable.

Implicit schemes are not unknown in wave propagation analysis in piles; one was used by Smith and Chow (1982). One reason for the preference for explicit schemes is the lower cost of computation (Randolph and Simons (1986)) and, of course, simplicity of construction. But implicit schemes deserve consideration for future wave equation algorithms.

5.4. Parasite Oscillations

One place where the limitations of numerical integration schemes manifest themselves is a phenomenon noted by Bossard and Corté (1983): "parasite oscillations" that are the product of the numerical integration scheme and not of the physical system.[3] Absent from (2) are the Heaviside step functions (Rausche, Goble and Likins (1985); Warrington (1997),) which denote a sharp discontinuity between what is ahead of the leading edge of the stress-wave (no stress) and behind it (stress.) This means that each blow of the hammer sends what amounts to a shock wave down the pile, as anyone familiar with pile driving will attest. These are a challenge to any discretization and numerical integration system. Given the difficulties inherent in the technique, how has Smith's and related routines been used successfully for the wave equation for piles in the forward solution, to say nothing of the inverse?

Taking the parasite oscillations problem first, the usual method of dealing with this is to add some dampening to the system that the physics of the system do not require. Although all real engineering materials contain some internal dampening, the dampening added is generally not based on material properties but upon the exigencies of the methodology. Also, in the very early years of the numerical solution of the wave equation, the piles analysed were typically onshore piles driven more or less to grade. Thus, there was dampening all along the shaft of the pile and at the toe as well; this enhanced the stability of the numerical integration. When these routines were applied to offshore conventional platforms, it was necessary to deal with long stretches of pile shaft undamped by soil.

Going back to the basic methodology, numerical methods are capable of producing "reasonable" results under some conditions even if they be strictly speaking inconsistent, unstable or by extension unconvergent. Many of these difficulties are masked by the imprecise nature of geotechnical engineering; the uncertainties of the input variables are a far larger source of error than the problems of the numerical method.

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[3]The formal term for these is "dispersive" effects of numerical integration.

Smith's method allowed an easy construction of a physical system with many non-linearities and variations in the basic stiffness of the components (and by extension the numerical system.) Although it is unwise to summarily jettison such a system, it is also unwise to uncritically accept it as a permanent *fait accompli*. If we can improve the numerical methods at hand, it should make detecting other problems of our system simpler and more meaningful, even with the uncertainties of the environment. As was the case at the dawn of numerical methods in wave propagation in piles, we can draw from other disciplines for assistance. For example, the demands of computational fluid dynamics have driven an improvement in numerical integration to minimize or eliminate these, and there is no good reason why this effort cannot be applied to pile dynamics.

One other thing that should be mentioned is that, although implicit methods can obviously improve performance, they are not the only way this can be accomplished. Either way, the possibilities of improving the predictive capabilities of either forward or inverse methods of the wave equation as applied to piles are considerable.

5.5. Method of Characteristics

One method employed for the analysis of wave propagation is the method of characteristics. This is described in some detail by Abbott (1966). According to Middendorp and Verbeek (2005), the concept of the method of characteristics was first proposed for driven piles with the soil resistance concentrated at the toe. In the course of the development of the HBG Hydroblok hammer, shaft resistance was added to the model and a practical method of analysing wave propagation in piles was developed (Voitus van Hamme et.al. (1974).) The method of characteristics is embodied in the TNOWAVE program. The method also has both forward and inverse application, and the latter is not restricted to TNOWAVE related applications, being used in CAPWAP-C. Horvath and

Killeavy (1988) were of the opinion that CAPWAP was improved by switching to the method of characteristics from the lumped-mass model.

In its simplest form, the method of characteristics divides up the pile into segments, as is the case with the other numerical methods. The difference comes in that the method of characteristics solves (3) for each segment and time step. Any other resistances or changes along the shaft or toe (soil resistance, change in pile impedance) are represented at the boundaries of each segment. Generally speaking, it is necessary to coordinate the segment length to the time step through the acoustic speed of the pile material. For uniform piles, this is fairly straightforward; where the pile has one or more changes in impedance, time step selection becomes more complicated.

The method of characteristics can thus avoid many of the stability problems inherent in the explicit methods; however, the comments related to (3) apply, to some extent, to the method of characteristics.

6. Finite Difference vs. Finite Element Methods

In approaching this subject for piles, there are two topics that get conflated in the discussion but which should be considered separately.

1. The method of discretisation and integration of the governing differential equations. This, strictly speaking, is the general distinction between finite difference and finite element methods. The former divides a field defined by a equation describing the physics into finite differences, while the latter treats each element as an individual system and then related the elements one to another in matrix form (the "direct stiffness matrix.")

2. The spatial method by which the hammer (sometimes,) pile and soil system are

discretised. "One-dimensional (1D) modelling" is what is usually associated with the wave equation for piles, with the familiar spring-dashpot system for the soil. "Two-dimensional (2D) modelling" (actually actually three dimensional, but axisymetrically implemented it can be considered as a two dimensional model) actually divides up the soil into elements, simulating a semi-infinite soil mass by special boundary conditions.

6.1. The Methods Themselves

The first topic concerns finite difference vs. finite element itself. Most engineers are familiar with finite-element methods for use in structural analysis, both static and dynamic. Finite-element analysis is also becoming more common in geotechnical engineering, finding its way into groundwater flow and soil strength calculations. Nevertheless, the Smith (1960) model used a finite-difference model of the system at hand. The wave equation was not alone: many other early analysis routines for deep foundations, such as the p-y modelling of lateral loads and t-z modelling of axial loads (Parker and Radhakrishnan (1975),) also used finite difference techniques. Why do these persist? There are two basic reasons.

The first is that the finite difference methods, and especially the explicit ones, were easier to set up mathematically. They did not necessarily require use of advanced linear algebra, and the complexities that this introduced into the methodology.

The second is that, for all of their advantages, finite elements have proven even more problematic than their finite-difference counterparts in producing parasite oscillations (Smith and Chow (1982); Deeks (1992); Warrington (1997)). Much of the reason for this, however, is due to the fact that the favoured method of time-integration has been the "Newmark β" Method, which does not always deal with the shock wave inherent in the process effectively. As is the case with finite-difference methods, changes in integration methodology have the promise of improving the stability of the integration.

6.2. Spatial Treatment

Generally speaking, finite element analysis of pile dynamics is associated with "2D modelling" of the pile-soil system. Up to now we have discussed "1D modelling;" with 2D modelling (Mabsout and Tassoulas (1994)) the soil immediately surrounding the soil is modelled for static and dynamic response to excitation. This association is not necessarily so; it is possible for finite element solutions to be either 1D or 2D, and in fact Mitwally and Novak (1988) developed a 1D finite element analysis solution which used the reults of 2D soil modelling to improve the 1D model. Such a model was also used by Danzinger et.al. (1996).

That being said, some discussion of 2D solutions are in order. In theory, these have been around for a long time (Smith and Chow (1982); Ebecken et.al. (1984),) but their implementation in practical use has not been widespread. They have even been developed for the inverse problem as well as the forward analysis (Masouleh and Fakharian (2008).) Although they have some obvious advantages over 1D methods, there are three reasons why 2D methods have not gained the acceptance of their 1D counterparts.

The first is that, up until now, most 2D methods have not modelled the hammer/cap/cushion system to any degree. To do so in finite elements is certainly possible but more difficult than it has been in finite differences, or more exactly lumped mass systems.

The second is that obtaining uniform results for the same input is more difficult with 2D methods than with 1D methods. This is mostly an acceptance issue. Codes and specifications generally "like" an easily reproducible calculation; this is one reason why the wave equation struggled for acceptance vs. the dynamic formulae. The growing use of other

2D and 3D finite element codes for analysis of geotechnical problems should facilitate this acceptance.

The third is that rheological issues are far more complex than with 1D codes (Pinto and Grazina (2008).) Geotechnical engineers are well aware of the difficulties in accurately quantifiying the response of soils, rocks, and materials in between for relatively simple problems such as bearing capacity and settlement. Additionally there is evidence that the relative wave speeds between pile and soil may render the complexity of rheological modelling in 2D an unproductive exercise (Héritier and Paquet (1986).) Unless the rheology embodied in the finite elements used for the soil can be shown to have a significant advantage in modelling the soil response, employing 2D modelling will not advance our predictive ability in pile dynamics.

7. Rheology

Discussion of rheology in pile dynamics for either 1D or 2D systems brings us to what is, in many ways, the "stickiest wicket" in pile dynamics. In addition to the usual problems of quantifying soil response, the nature of pile dynamics, especially in the inverse problem, brings an entirely new dimension to the problem.

7.1. Overview of the Problem

To understand why this is so, we start by looking at Figure 1, which shows a t-z model of a deep foundation. The idea of t-z analysis is to essentially replicate a static load test–and by extension a pile under service load–numerically. In accepting this kind of methodology, we implicitly accept another concept: that the whole idea of "bearing capacity" as we understand it for shallow foundations doesn't really apply to deep foundations. Bearing capacity implies failure due to exceeding the resistive capacity of a shear surface. Bearing capacity failure, like slope stability failures subject to a similar failure mode, tends to be catastrophic. Such failures are mercifully rare with deep foundations and, when they do occur, usually are caused by phenomena which are different from classic bearing capacity failure. Deep foundations are designed to keep settlements for given loads to a level the structure can tolerate; a t-z analysis (like a static load test) allows us to quantify the load-settlement characteristics of a deep foundation and in doing so to determine their allowable loading.[4]

That being said, let us look at Figure 2, which is a "classic" wave equation model Figure 1 depicts the use of purely non-linear models for the response of the soil to pile deflection. This is in accordance with the idea, for example, of the hyperbolic model of soil response such as described in Duncan and Chang (1970). If we look at Figure 2, we see that the displacement response of the soil is modelled with an elasto-plastic model. The application of a hyperbolic model to a dynamic analysis of pile driving was considered by Nath (1990), who concluded that the elasto-plastic model was sufficently close not to introduce the complexities of the former into the analysis.

Yet another–and more imortant–difference between the two is the absence or presence of velocity-dependent dashpots. Obviously, Figure 1 leaves these out, as the whole concept of a static load test is to load the pile in such a way that, among other things, there are

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[4]This issue also highlights another thing to keep in mind concerning static load test correlations: the bearing capacity that is derived from a static load test is dependent upon the failure criterion being applied. Although certain criteria (such as Davisson's) are used extensively, differing criteria can produce differing bearing capacity results. This needs to be kept in mind when reviewing correlations between static load test data and the results of dynamic testing.

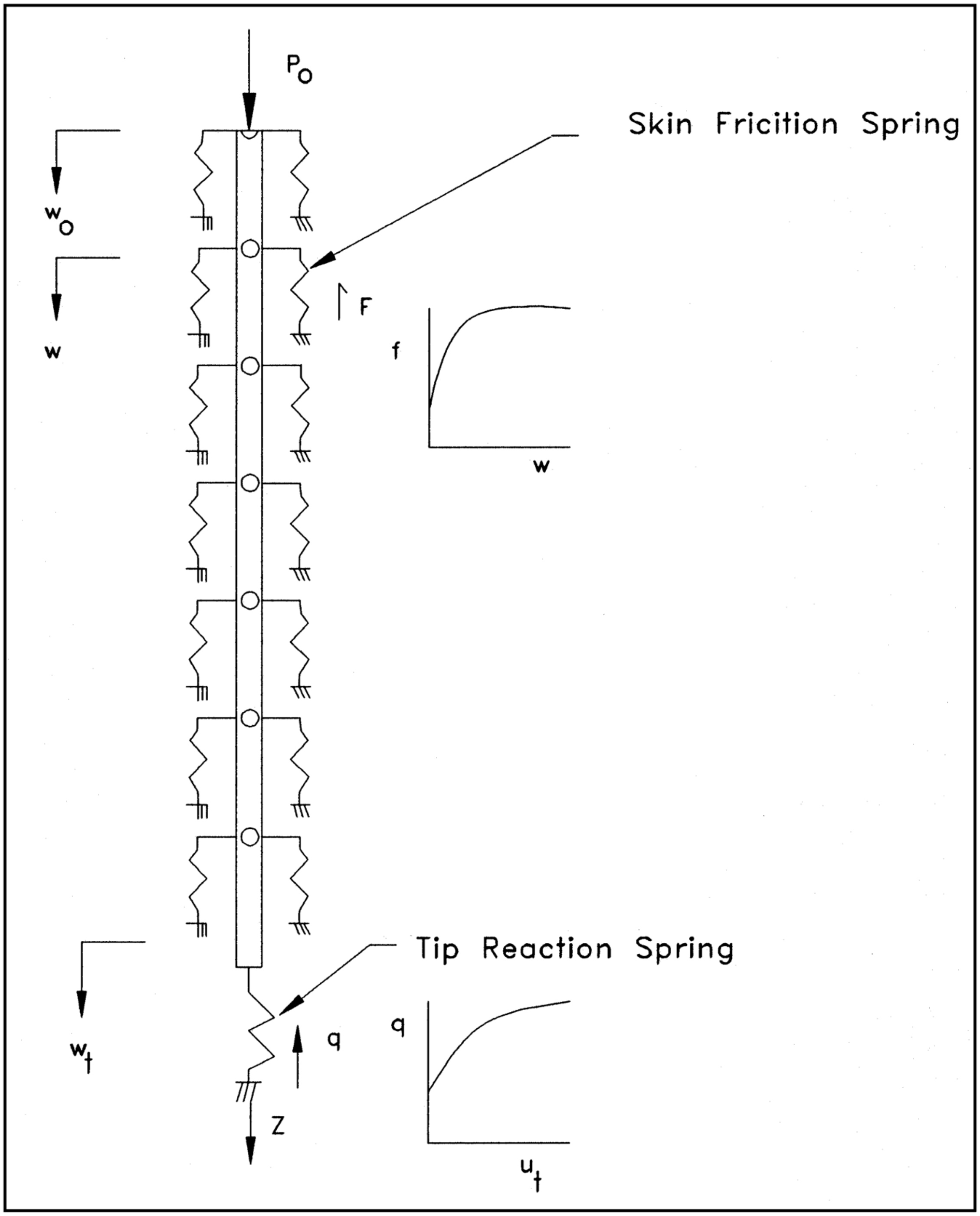

Fig. 1.— t-z Diagram for Statically Loaded Pile (Mosher and Dawkins (2000))

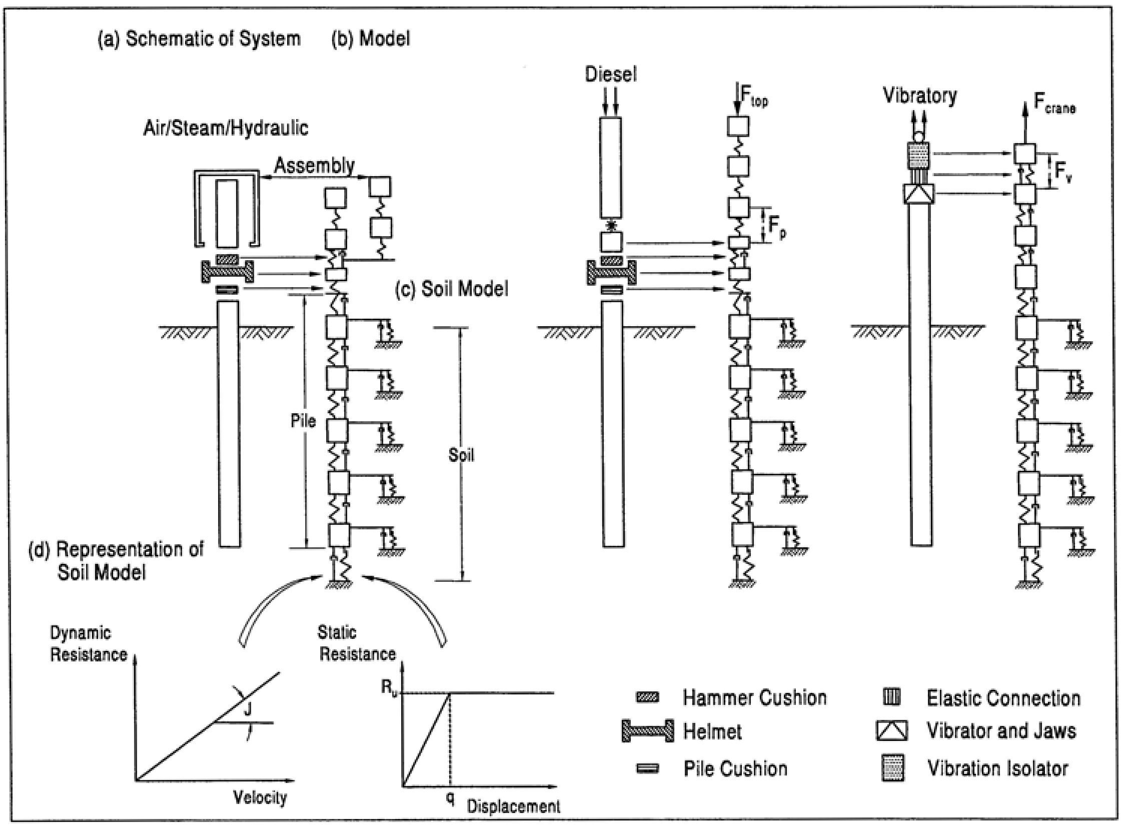

Fig. 2.— One-Dimensional Wave Equation Model for Various Driving Equipment (after Hannigan et.al. (1997))

no velocity-dependent results from the test. With dynamic prediction and testing depicted in Figure 2, velocity dependent results are inevitable, be they caused by purely viscous dampening, (mostly) radiation dampening, or velocity dependent responses from other sources.

Comparing these two pile models brings us to the ultimate objective of pile dynamics: to separate the velocity-dependent dampening that is experienced during dynamic loading from the static response of the soil. Assuming the loading rate does not affect the latter–a significant assumption–removal of the former leaves us with the information required to run a t-z type of analysis and thus estimate the load-displacement characteristics of the deep foundation. This separation of static and dynamic resistance is more easily seen in methods where the distributed mass and elasticity of the foundation does not complicate the analysis, such as Statnamic analysis (Warrington (2007).) An ideal result would be the construction of a pile top axial load-deflection relationship from dynamic testing that could be compared profitably with one derived from a static load test.

7.2. Difficulties in the Solution

Getting to this, however, is not as simple as it looks. The easiest way to see this is to consider the pile top response for a uniform semi-infinite pile subject to a pile top force $F_0(t)$ and governed by (3). The solution to this is (Warrington (1997))

$$u(0,t) = \frac{1}{Z}\int_0^t e^{-bt} I_0\left(\sqrt{(b^2-a)(\tau^2)}\right) F_0(t-\tau)\,d\tau \tag{8}$$

In this case Z is the pile impedance and I_0 is a Bessel function (Bowman (1958)), which appear frequently in any analytical solution of this kind.

Our objective is to determine the linear static coefficient a and the viscous coefficient

b. The pile top force $F_0(t)$ can be determined from the strain gauges. The pile top displacement can be determined either a) directly through a high-speed theodolite (SIMBAT) or b) through double integration of the accelerometer data (CAPWAP). Z is a property of the pile cross-sectional configuration and material properties.

Even in this relatively simple form, (8) is difficult to solve analytically. The simplest way for real force-time curves is to do so implicitly, using a root-finding method. Doing that, however, does not avoid the simple fact that we have two unknowns (a and b) and only one equation. It is thus impossible, using data from the pile top, to separate the static and dynamic components of the pile resistance.

To put this rather surprising result into perspective, let us write (3) in a broader form, thus

$$c^2 \frac{\partial^2}{\partial x^2} u(x,t) = \frac{\partial^2}{\partial t^2} u(x,t) + a(x)\, u(x,t) + 2\, b(x) \frac{\partial}{\partial t} u(x,t) + d(x) \frac{\partial}{\partial x} u(x,t) \qquad (9)$$

where we make two important additions. The first is that the functions of elasticity, viscocity, etc., along the shaft can vary with x. This is an important addition because of of the things we seek in dynamic testing is the distribution of resistance along the pile shaft, which is in turn based upon the distribution of elasticity and viscocity. The second is that we add a strain term to the equation.

There is no dispute that it is possible, using data from one "end" of a system governed by (9) to determine the properties of the system. However, a quick survey of inverse methods such as Gelfand and Levitan (1951), implementations of same such as Ning and Yamamoto (2008) or methods such as boundary control, we note that either $a(x) = 0$ or $b(x) = 0$. The inclusion of both creates the situation we have in (8). This problem was recognized at the beginning of modern dynamic analysis (Rausche et.al. (1972).)

This does not mean that all hope is lost. We can solve this problem if we can establish a relationship between $a\,(x)$ and $b\,(x)$, or to be more precise if they are functions of each other. Once this is established (8) becomes one equation in one unknown. What this means is that the portion of the resistance during driving we ascribe to the dynamic portion depends upon our assumptions regarding the rheology of the soil and not the inverse methodology at hand. If all other components of the inverse methodology are correct, then the key decision in the process comes with the assignment of soil properties before the reduction of data.

One thing that might alter this situation to some extent is the presence of non-linearity in both the elastic and viscous portions of the resistance. This is, in fact, the approach implied by Rausche, Goble and Likins (1985), where they state that "...the uniqueness of the CAPWAP resistance distribution is proven under the assumption of an ideal plastic soil behaviour." Eliminating the elasticity of the system, however, makes the soil model inconsistent with those used in the forward methods such as are depicted in Figure 2 (Danzinger et.al. (1996).) And the selection of rheology matters here as well. Different rheologies, even those of "similar" type, treat non-linearity differently; one only needs to compare that of Smith (1960) to that of, say, Randolph and Simons (1986) and Corté and Lepert (1986) to understand this.

7.3. Uncertainty in the Significance

The ability to predict the distribution of soil resistance along the shaft of the pile–and the distribution between the shaft and the toe–has been considered one of the important results of dynamic testing. We have shown that this result can be obscured by the nature of the problem, but we must now ask another question: how significant are these difficulties? What, for example, would be the problem if we could only predict the overall resistance of the pile and not be so concerned with how it varies within the system?

The simple answer to this lies in another well-known difficulty of dynamic testing: set-up effects in soils. If we have a relatively homogeneous soil profile to drive into, with a uniform set-up factor, then the distribution of resistance along and at the end of the pile may not be as significant as it seems. But, as is often the case in geotechical engineering, we have soils that vary both in kind and in resistance characteristics (loose vs. dense, soft vs. stiff) then knowledge of this distribution becomes critical. Inverse methodolgy usually deals with this by measuring the resistance at the end of driving.

Research on the sensitivity of this distribution (both in terms of pure resistance and factors such as dampening) is not copious. One interesting study is that of Meseck (1985), who examined this using WEAP. Running parameteric studies, he concluded that factors such as hammer and cushion configuration and soil dampening were critical, while others such as quake and–most importantly–the distribution of resistance along the shaft and/or the distribution of resistance between shaft and toe were not as critical, as were variations in pile length and elasticity.

If precision in the determination of resistance distribution parameters is not critical in the overall drivability–and by extension capacity–analysis of a pile, then some of the problems we have described above may not be as important as they seem theoretically. This especially relates to the uniqueness problem re the distribution of the shaft and toe resistance. On the other hand, the results of Meseck (1985) emphasise the importance of the proper estimate of damping parameters, which brings us back squarely to the issue of separating static from dynamic resistance.

8. Conclusions

The analysis of wave mechanics in piles during their installation–or in some cases as a test mechanism–is an important advance in the design, installation and verification of driven pile and other deep foundations. Nevertheless the methodologies in place, like any other scientific and engineering discipline, are subject to improvement. Although much attention is given to fine-tuning to match static load tests, there are some fundamental issues of the methodology that, if properly investigated and implemented, could move the art of deep foundation design forward in a more significant way.

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