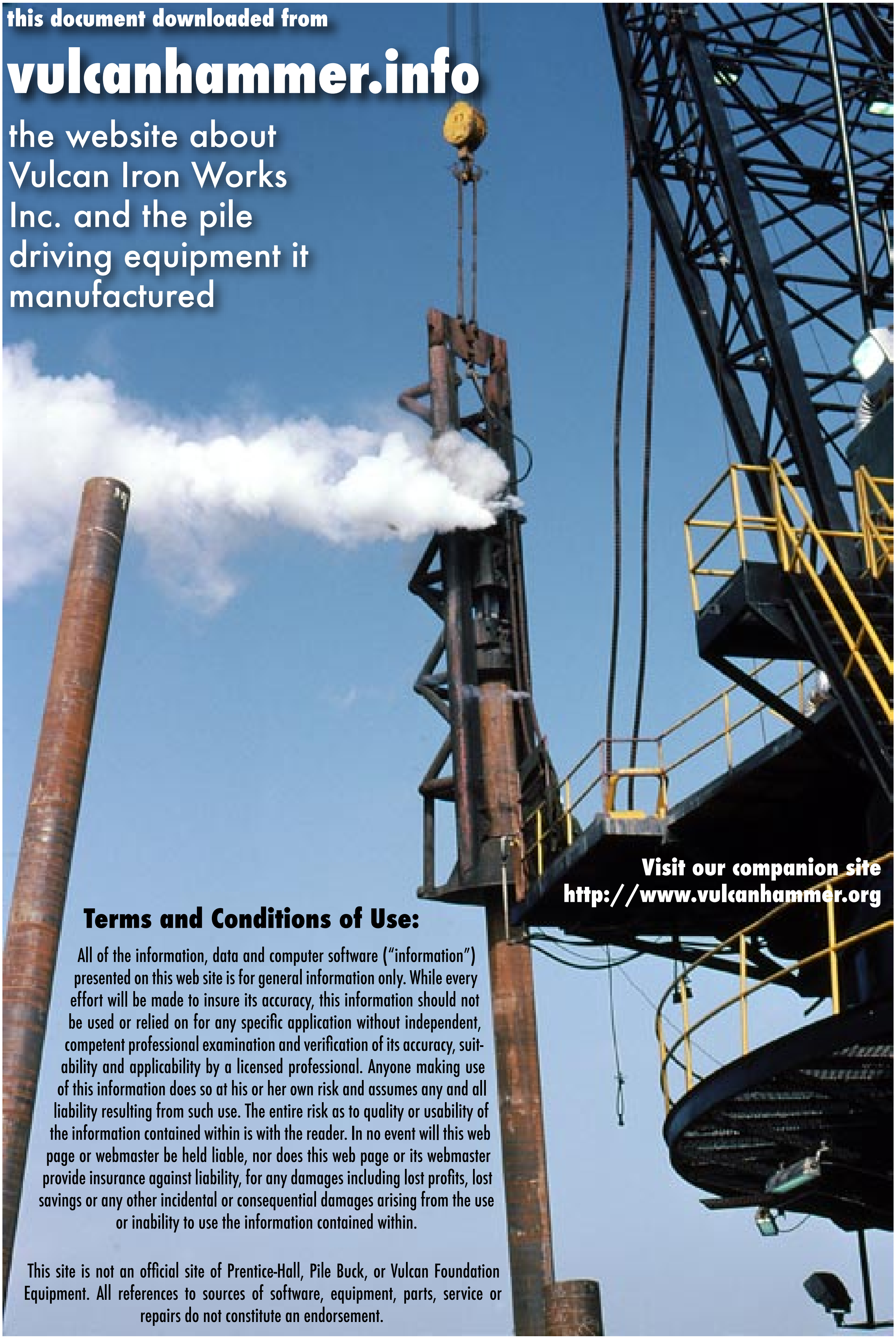

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TECHNICAL NOTE

NUMERICAL APPROXIMATIONS IN PILE-DRIVING ANALYSIS

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SUMMARY

A finite-difference scheme widely used in analysis of pile driving is examined. the scheme is shown to yield peculiar results in simple symmetric impact problems and to inherently possess an imprecise definition of energy balance. A modified difference scheme is suggested and comparisons of the two methods are shown.

INTRODUCTION

The use of wave mechanics to analyse the response of pile-soil systems has gained widespread acceptance in recent years. A one-dimensional idealized wave front, generated by impact of a pile hammer, is traced as it propagates down the pile, eventually reflecting from the pile tip. Soil resistance is modelled both through side friction along the length of the pile and at the tip. The approach is not only a more realistic approximation to the actual physical phenomena ocurring in a pile driving, but is considered to possess several additional advantages over conventional dynamic formulae. Prominent among these advantages are

1. the ability to predict whether or not the pile can be driven using the proposed pile–pile hammer combination.
2. the ability to estimate the ultimate load capacity of the pile before driving, and
3. the ability to predict the stress magnitude in the pile due to hammer impact.

The first mention of wave mechanics associated with pile driving was evidently due to Isaacs[1] in 1931. He gave a lengthy development of the problem including a number of assumptions concerning the shape of the wave form and the attenuation of the wave with distance of propagation. These assumptions were made in order to obtain analytic solutions, the equations of motion being too complicated for direct solution. Beginning in 1950, the concept was revived by E. A. L. Smith using numerical methods. Smith constructed a simple lumped-parameter discretization of the pile and pile hammer and employed a relatively simple constitutive relation for the soil resistance. His work is summarized in Reference 2. Smith's work generated considerable interest and has been the subject of several recent reports.[3–6] In 1974, his method was reproduced in a widely distributed textbook.[7]

Research subsequent to Smith's work has been directed toward either validating his concepts through comparisons with actual pile-driving situations or in attempting to improve constitutive representations of pile hammers, cushions, soil resistance, etc. In all cases, Smith's

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lumped-parameter analogy, as well as his finite-difference formulation, have been used. Yet a simple examination of his difference scheme raises interesting questions. It is well known that, unlike most numerical approximations, one-dimensional waves in a linearly elastic body may be exactly represented by a finite-difference approximation. Letting c denote the velocity of propagation in the material, exact solutions are obtained when the time step of integration, Δt, is set equal to $\Delta x/c$, Δx being the spatial distance separating adjacent lumped masses. In effect, the space–time finite difference mesh nodes fall exactly on the characteristic curves of the analytic solution, and the exact solution is obtained. Yet, if the soil resistance in Smith's scheme is set to zero, and the critical time step is employed, the exact solution to a simple impact problem is not found. Instead, the difference equations yield an oscillating wave form whose mean value is the exact solution.

In this paper we suggest an alternative difference scheme for the pile-driving problem. The alternative method is explicit (in that all quantities at any new time step depend only on values at the preceding time) and thus is as easily programmed as Smith's scheme; yet it appears to yield smoother and more accurate solutions than Smith's method, and, in the special case where $\Delta t = \Delta x/c$, yields exact solutions to elastic problems.

SMITH'S METHOD

Smith's difference equations are derived from an intuitive application of the momentum balance equation to systems of discrete masses separated by linear springs. The pile is broken into individual lumped masses, m_j. Its elasticity is represented by linear spring constants, k_j. The difference equations determine the displacement, u_j^n, velocity, v_j^n, and acceleration, a_j^n, of each mass. Here, the superscript n denotes the functional value at time $n\Delta t$ while the subscript j identifies the appropriate mass, m_j. Momentum balance of the jth mass is expressed by

$$v_j^{n+1} = v_j^n + a_j^{n+1}\,\Delta t \tag{1}$$

where the mass acceleration is

$$a_j^{n+1} = (F_{j-1}^{n+1} - F_j^{n+1} - R_j^{n+1})/m_j \tag{2}$$

Here, F_j^{n+1} is the force in the spring below the jth mass

$$F_j^{n+1} = k_j(u_j^{n+1} - u_{j+1}^{n+1}) \tag{3}$$

The quantity R_j^{n+1} represents the frictional resistance of the soil surrounding the pile. It is given by one of several special constitutive relations which depend on the displacements u_j^{n+1}. Finally, a kinematic description of the motion of the jth mass is given by

$$u_j^{n+1} = u_j^n + v_j^n\,\Delta t \tag{4}$$

Given initial conditions on u_j^n and v_j^n, the displacement u_j^{n+1} is calculated from equation (4). Equation (3) is then used to determine the pile forces. The soil resistance is also calculated at this stage. Next, equation (2) is used to find the acceleration, a_j^{n+1}, and finally equation (1) yields the new mass velocities, v_j^{n+1}. The scheme is explicit, in that all quantities at time $(n+1)\Delta t$ depend only on values at time $n\Delta t$. It can also be shown that the usual numerical stability condition

$$\Delta t \leq \sqrt{\left(\frac{m_j}{k_j}\right)} \tag{5}$$

applies. This criterion is derived from the Courant condition, $\Delta t \leqslant \Delta x/c$ (for example, see Reference 8).

In the special case where the soil resistance, R^n_j, is set to zero, Smith's equations should describe the response of a linearly elastic material in uniaxial deformation. We may further simplify the equations by making all the discrete masses equal, so that $m_j = m$. and all spring constants equal, so that $k_j = k$, thus modelling a homogeneous material. A simple illustration of the unusual behaviour of Smith's scheme is then given by the case of a symmetric impact between identical materials. Letting the impact occur at the pth mass, the appropriate initial conditions are

$$u^0_j = 0 \quad \text{for all } j \tag{6a}$$

$$v^0_j = \begin{cases} 1 & \text{for all } j < p \\ 0 & \text{for all } j \geqslant p \end{cases} \tag{6b}$$

The analytic solution of this problem is well known. A step change in velocity, equal to one-half the impact velocity, propagates into the stationary material, while a similar velocity decrease propagates back into the impacting material. For convenience, we set $k/m = 1{\cdot}0$. Then using the maximum allowable time step,

$$\Delta t = \sqrt{\left(\frac{m}{k}\right)} = 1{\cdot}0$$

Smith's scheme yields results shown in Table I. Only the first four cycles of the integration are shown in the table, yet it is clear that the impact is propagated as a saw-tooth wave form, the mass velocities alternating between 1·0 and 0. the correct mass velocities are shown in parentheses.

Table I. Pile element velocities in symmetric impact

Time step	Mass number							
	$p-4$	$p-3$	$p-2$	$p-1$	p	$p+1$	$p+2$	$p+3$
0	1 (1)*	1 (1)	1 (1)	1 (1)	0 (0)	0 (0)	0 (0)	0 (0)
1	1 (1)	1 (1)	1 (1)	0 ($\frac{1}{2}$)	1 ($\frac{1}{2}$)	0 (0)	0 (0)	0 (0)
2	1 (1)	1 (1)	0 ($\frac{1}{2}$)	1 ($\frac{1}{2}$)	0 ($\frac{1}{2}$)	1 ($\frac{1}{2}$)	0 (0)	0 (0)
3	1 (1)	0 ($\frac{1}{2}$)	1 ($\frac{1}{2}$)	0 ($\frac{1}{2}$)	1 ($\frac{1}{2}$)	0 ($\frac{1}{2}$)	1 ($\frac{1}{2}$)	0 (0)
4	0 ($\frac{1}{2}$)	1 ($\frac{1}{2}$)	0 ($\frac{1}{2}$)	1 ($\frac{1}{2}$)	0 ($\frac{1}{2}$)	1 ($\frac{1}{2}$)	0 ($\frac{1}{2}$)	1 ($\frac{1}{2}$)

*Correct velocities shown in parentheses.

It is also interesting to investigate the implications of Smith's formulation on the balance of energy in the medium. In the absence of thermodynamic effects, conservation of energy should follow directly from momentum balance. For the case of no soil resistance, the increment of work done on mass m_j between times $n\,\Delta t$ and $(n+1)\,\Delta t$ is

$$\mathrm{d}W^{n+1}_j = (F^{n+1}_{j-1} - F^{n+1}_j)(u^{n+1}_j - u^n_j)$$

Forces may be eliminated from this expression using equations (2) and (1). Displacements are eliminated through equation (4). This results in

$$\mathrm{d}W^{n+1}_j = m_j(v^{n+1}_j - v^n_j)v^n_j \tag{7}$$

fully specifying $F_j^{n+\frac{1}{2}}$ in terms of quantities known at time $n\,\Delta t$. Special constitutive relations for $R_j^{n+\frac{1}{2}}$ which depend on $u_j^{n+\frac{1}{2}}$ may be similarly determined. In essence, equations (1), (2), (3) and (4) are now replaced by equations (9), (10), (16) and (13) respectively. The stability criterion (5) applies equally well here,* and, because of the approximation inherent in equation (15), the modified scheme is explicit. Although equation (16) may appear at first glance to represent a visco-elastic material, this is not the case. The velocities appear only to correctly centre the displacements at time $(n+\frac{1}{2})\,\Delta t$.

Energy is exactly conserved by the modified equations. The increment of work done on mass m_j during the time interval Δt is now given by

$$\mathrm{d}W_j^{n+1} = (F_{j-1}^{n+\frac{1}{2}} - F_j^{n+\frac{1}{2}} - R_j^{n+\frac{1}{2}})(u_j^{n+1} - u_j^n) \tag{17}$$

Using equations (10) and (13) here yields

$$\mathrm{d}W_j^{n+1} = m_j a_j^{n+\frac{1}{2}}(v_j^n + \tfrac{1}{2}a_j^{n+\frac{1}{2}}\Delta t)\,\Delta t \tag{18}$$

Eliminating the accelerations through equation (9) then gives

$$\mathrm{d}W_j^{n+1} = m_j(v_j^{n+1} - v_j^n)\left(\frac{v_j^{n+1} + v_j^n}{2}\right) \tag{20}$$

which is precisely the change in kinetic energy of the mass m_j during the time interval.

Setting $R_j^{n+\frac{1}{2}} = 0$, $m_j = m$, and $k_j = k$, we may again consider the symmetric impact problem. As before, we take $\Delta t = \sqrt{(m/k)} = 1{\cdot}0$. Then combining equations (9), (10), (13) and (16), the governing equations become

$$v_j^{n+1} = \tfrac{1}{2}(v_{j-1}^n + v_{j+1}^n) + u_{j-1}^n - 2u_j^n + u_{j+1}^n \tag{21}$$

$$u_j^{n+1} = \tfrac{1}{2}(u_{j-1}^n + u_{j+1}^n) + \tfrac{1}{4}(v_{j-1}^n + 2v_j^n + v_{j+1}^n) \tag{22}$$

Using the initial conditions given in equations (6a) and (6b), a simple calculation shows that the exact solution is obtained for symmetric impact.

COMPARISON OF METHODS

In order to compare the modified scheme with that of Smith, we consider two problems. First, the simple symmetric impact problem was run with both methods employing values of Δt between one-half the critical value and the critical value. Second, both methods were used to calculate a problem with realistic pile and soil parameters. The first comparison is more interesting from an analytic standpoint, since the exact solution is known. The second example is also of interest, however, as it illustrates the magnitude of discrepancies which may arise in typical calculations solely due to altering the difference scheme.

Velocity profiles for the symmetric impact problem with values of $\Delta t\sqrt{(k/m)}$ ranging between 0·5 and 1·0 in increments of 0·1 are illustrated in Figure 1. Smith has recommended[2] that a value of 0·5 be used in practical calculations. It is evident from the figure that both methods yield similar results at this value. As the time step is increased, oscillations in Smith's method grow accordingly, culminating in the saw-tooth wave form at $\Delta t\sqrt{(k/m)} = 1{\cdot}0$. The modified method yields nearly uniform results at all time step values except 1·0, where the exact solution is found. It seems clear that the modified method is capable of running at larger values of Δt without sacrificing accuracy, and thus may be more economical for long or repeated calculations.

* This is not surprising, since both difference schemes represent the same materials with identical geometry.

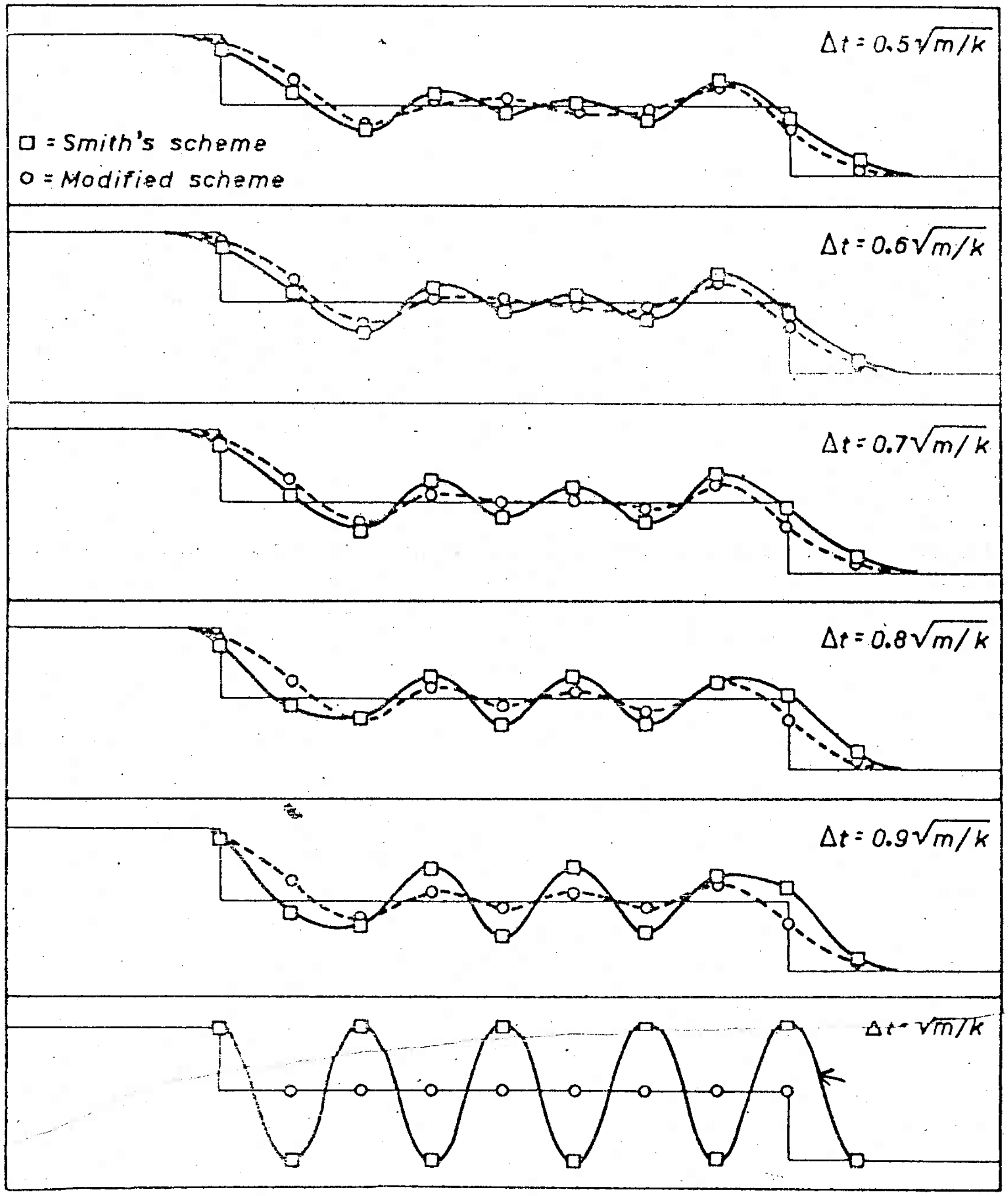

Figure 1. Velocity wave forms for symmetric impact

The second example problem was run by Phelan.[9] Both Smith's and the modified methods were used. In both calculations exactly the same pile and soil parameter values were employed. The hammer, 7×10^3 kg in mass, was divided into two elements and given a stiffness, k, of $14\cdot1\times10^3$ kN/mm. The pile consisted of eleven elements, each $6\cdot67\times10^3$ kg in mass, with an assumed stiffness of $2{\cdot}67\times10^3$ kN/mm. Ten of the eleven pile elements were resisted by an ultimate side friction of 800 kN. The soil quake (see Smith[2] for a discussion of this parameter) was taken as 3·0 mm. A point resistance of 8×10^3 kN was applied at the pile tip. The hammer impact velocity was taken as 10 m/s, and a time step of $2{\cdot}5\times10^{-3}$ s (equal to one-half the critical value) was employed. Velocity wave forms for both methods at $n = 13$ are shown in Figure 2. Although similar results are found near the wave front, the solutions show considerable difference near the head of the pile. Further results from both analyses are shown in Table II. The modified scheme gives a permanent set 21 per cent less than that obtained by Smith's method. The maximum pile force is reduced by nearly 17 per cent.

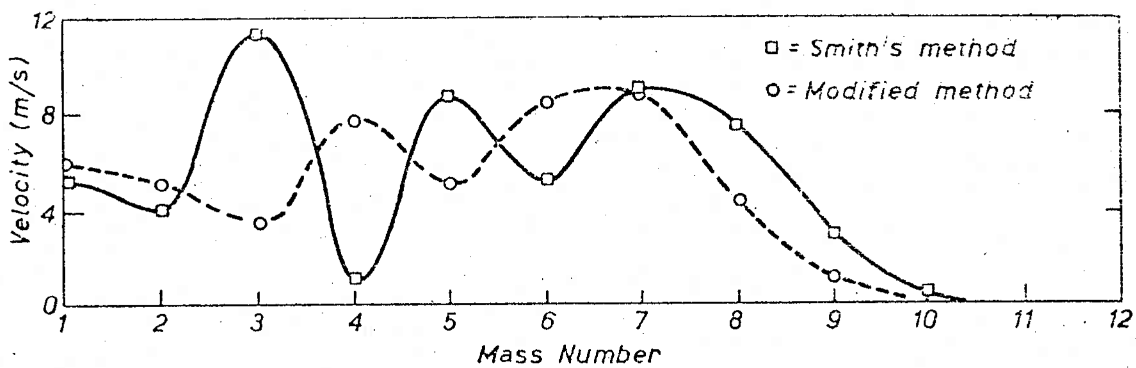

Figure 2. Velocity wave forms for typical pile parameters

Table II. Comparison of Smith's and modified methods

Quantity	Smith's method	Modified method
Maximum force in pile	18×10^3 kN	15×10^3 kN
Maximum force in hammer	37×10^3 kN	16×10^3 kN
Permanent set	10·0 mm	7·9 mm

CONCLUSIONS

From the standpoint of the numerical analyst, the modified difference scheme presented here offers two attractive advantages over Smith's method. The first of these is that energy balance in the pile is formulated in a precise manner, consistent with the difference approximation of momentum balance. The second advantage is that the modified scheme yields better approximations for impact problems which have analytic solutions. In essence, the only true test of accuracy for any difference approximation is comparison with closed-form solutions. In this regard, the modified scheme is superior, especially when values of Δt close to the stability limit are used. In problems involving realistic soil resistance where no analytic solutions are available, we can only point out that the two methods yield different results. Whether one or the other is, in fact, closer to reality is not a matter for discussion here. The closeness of approximation to physically real conditions depends on the degree to which the fundamental mathematical model (i.e. one-dimensional wave propagation) represents the true physical situation. Once the mathematical representation has been established, the problem of which numerical approximation to use is purely a matter of (1) convenience of calculation, and (2) how well the difference scheme compares with available analytic solutions. The modified scheme involves no more calculations than does Smith's method, and, for impact problems, yields results more consistent with closed-form solutions.

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