DEVELOPMENT AND POTENTIAL OF THE WAVE EQUATION IN CLOSED FORM AS APPLIED TO PILE DYNAMICS[1]

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Abstract

This paper is a review and summary of the efforts made to develop a closed form solution of the wave equation as applied to driven piles. It discusses the early development of solutions and includes discussion of such topics as semi-infinite piles, solutions using both Fourier series and the method of images, and solutions specific to vibratory hammers. Results, advantages, and limitations of each of these methods are discussed. The rationale for the use of closed form solutions as opposed to the numerical ones is set forth. The paper concludes with a discussion of the possibilities of future research and sets forth the requirements for making this research successful.

The Wave Equation In General

The classical one dimensional wave equation is given by the formula

$u_{tt}(x,t) = c^2 u_{xx}(x,t)$ .... (1)

where $u(x,t)$ = displacement, m
c = a constant
x = distance along the length of the rod, m
t = time, seconds

For longitudinal vibrations, the constant c is the acoustic speed of the material of the bar, given by the equation

$c = \sqrt{(E/\rho)}$ .... (2)

where c = acoustic speed of the material, m/sec
E = Young's modulus of the material, Pa
ρ = density of the material, kg/m^3

Equation (1) is a hyperbolic, second order partial differential equation. Using the technique of separation of variables, the most common solution of this equation is

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$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos(n\pi ct/L) + B_n \sin(n\pi ct/L)) \sin(n\pi x/L) \quad (3)$$

where L = length of the rod or string, m

and where the coefficients A_n and B_n are defined as

$$A_n = \int_0^L (2/L) f(x) \sin(n\pi x/L)\, dx \quad (4a)$$

and

$$B_n = \int_0^L (2/(n\pi c)) g(x) \sin(n\pi x/L)\, dx \quad (4b)$$

where A_n, B_n = orthogonal constants of integration
$f(x) = u(x,0)$
$g(x) = u_t(x,0)$

A detailed discussion of this subject is given in Petrovskii (1967); a history of its derivation is given by McCurdy (1993). As a result of the separation of variables, Equation (3) is essentially the solution of the Sturm-Liouville problem and is the sum of an infinite series of orthogonal eigenfunctons. The values for the coefficients A_n and B_n of this equation depend upon the initial conditions, i.e. the distribution of initial displacement and velocity along the axial bar.

Unfortunately Equation (3) is not directly applicable to pile driving because the boundary conditions of the solution are

$$u(0,t) = u(x,t) = 0 \quad (5)$$

that is to say the ends are fixed. Obviously this solution is of little value for pile driving, as a) the excitation of the system is coming at one of the boundaries, b) the central object of wave equation analysis is to analyse the movement of the pile, which is precluded by these boundary conditions, and c) no account is taken of any effects along the length of the pile.

It follows from this discussion that, although pile driving affords the possibility of using closed form one dimensional wave equation analysis, a useful solution to this problem for driven piles will involve some significant modification to both Equations (1) and (3).

Timoshenko's Presentation

No overview of the wave equation is complete without discussion of the presentation by Timoshenko and Goodier (1970) on both the propagation of longitudinal waves and impact of uniform bars. This discussion has been very influential in the stimulation of thinking on this subject. They establish the relationship between stress and velocity as

$\sigma_x = \rho c u_t$ ....(6)

where σ_x = stress in the bar, Pa

ρ = density of the pile material, kg/m^3

They go on to develop the equations of response for a bar fixed at one end and impacted by a rigid mass at another. In the course of this development they make several important observations:

1) In a stress wave moving down a bar, the energy contained in the stress wave is evenly divided between kinetic energy and potential (i.e., elastic compression) energy.

2) Stress waves are subject to superposition.

3) When a rigid body strikes a bar, for a given bar material the initial stress is a linear function of the velocity. Ignoring reflections from the fixed end, the stress in the bar at the end of impact is given by the equation

$\sigma = \sigma_0 e^{-(t\sqrt{(\rho E)}/M)}$ ....(7)

where σ_0 = initial stress @ impacted end @ t=0 = $\rho c v_0$

v_0 = velocity of mass or ram at impact, m/sec

M = mass of ram, kg.

4) At the end of the discussion, they make an important observation that is very much a part of pile driving (see Yamagata and Seto (1985)):

> The theory of impact developed above is based on the assumption that contact takes place over the whole surface of the end of the rod. This condition is difficult to realize. Elaborate precautions are necessary to ensure accurate plane ends and accurate alignment of the rods and to minimize the effect of the air film trapped between the ends. Then the observed wave propagation agrees well with the elementary theory.

Although Timoshenko's presentation is very important conceptually, he assumes a) no resistance along the side of the bar, and b) the end opposite from the impact was considered fixed, both problems of Equation (3).

Definition of a Closed Form Solution

Because of the wide variety of solutions for differential equations, it is necessary to define what kinds of solutions are "closed form." For the purposes of this paper, a closed form solution of the wave equation for a pile is one where the solution of the governing differential equation is integrated directly, whether to an equation or system of equations or to an infinite series, without resorting to numerical methods.

In order to attempt to make sense out of the work that has been carried out, four categories of solutions will be considered: 1) solutions using the method of images, 2) Fourier series types of solution, 3) solutions specific to vibratory hammers and excitation by a single frequency, and 4) solutions relating to piles of semi-infinite length. Because of its position in the development of the technology, the work of Isaacs (1931) is considered as well. In some cases the formulas reproduced have notation changes to arrive at a consistent notation system in the paper.

Isaacs' Work

The first observation of stress waves in piles is given by Isaacs (1931). The dynamic formulae had been developed primarily with timber piles in mind; with the growing usage of concrete piles, it became apparent that, because of the length and properties of timber pile, the dynamic formulae (with their assumption that the pile is a rigid mass) would not be sufficient for concrete piles. Isaacs started out by reviewing the dynamic formulae. Part of his review included a discussion of the factor of safety, where he makes a statement that is still relevant:

> It should be remembered, however, that these are not true factors of safety, but include a "factor of ignorance." The author suggests that when the ultimate resistance of any pile has been determined, in fixing the factor of safety...the most unfavourable conditions possible in the supporting strata should be judged (the range of conditions possible being narrowed with better knowledge of the subsurface conditions and of the possibility of disturbance from extraneous sources) and a proportion of the factor of safety -- a "factor of ignorance" -- then allowed in respect to these possible conditions, the manner of determining the ultimate load, and the type of loading to be borne. The remaining proportion of the factor of safety -- or *true* margin of safety -- should be approximately constant for all classes of loading and foundation conditions involving the same value of loss in case of failure; and the overall factor of safety...will then be equal to the product of the true factor of safety with the "factor of ignorance." (p. 305)

After this, he describes an experiment where rods are impacted against each other in a pendulum set-up. As the rods were lengthened, the behaviour of the rods deviated more and more from Newtonian impact theory.

He then went on to develop an integration technique that is best described as a semi-graphical one. He developed a mathematical model based on the successive transmission and reflection of waves (similar in principle to the method of images described below.) A sample solution is given in Figure 1, in this case showing multiple impacts. He then constructed a drafting machine to draw out the solution, a diagram of which is shown in Figure 2.

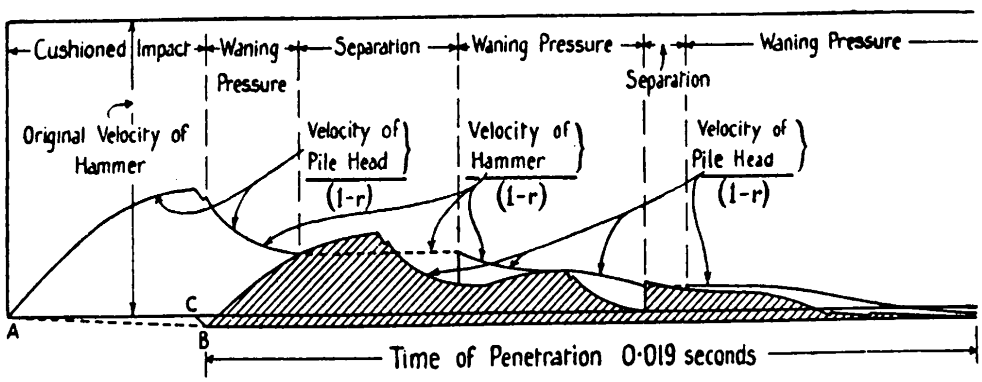

Figure 1 Graphical Solution of the Wave Equation (after Isaacs (1931))

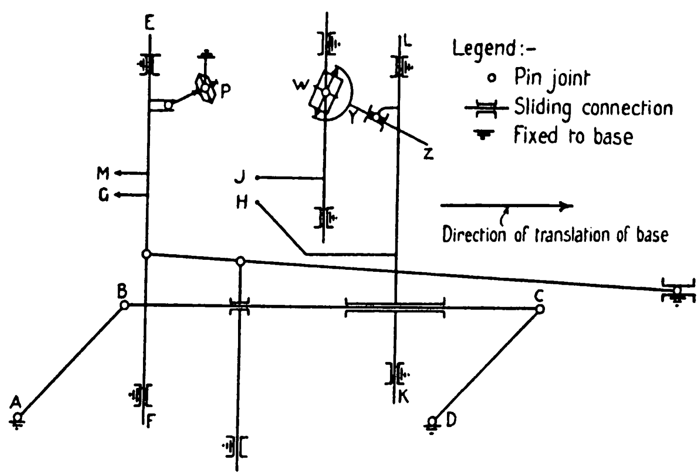

Figure 2 Graphical Machine for Wave Equation Solution (after Isaacs (1931))

He was then able to solve for the stresses and displacements of the pile during driving. Isaacs developed a set of formulae and charts to make his results accessible for the analysis of piles.

In the course of the investigation, Isaacs dealt with a number of questions that would become central to stress wave analysis of piles, including tension stresses in concrete piles, the effect of ram weight (he concluded that to a point a heavier ram reduced tension stresses,) and the effect of cushion material stiffness and drive cap weight.

Issacs' work also revealed the computational complexity of stress wave analysis, a complexity that insured the dominance of dynamic formulae in pile analysis (with all of their serious limitations) for another half century.

Method of Images

It can be shown that Equation (1) can be solved in the form

$u(x,t) = f(x-ct) + g(x+ct)$...(8)

where f(x-ct), g(x+ct) = functions of x and t which possess continuous second derivatives

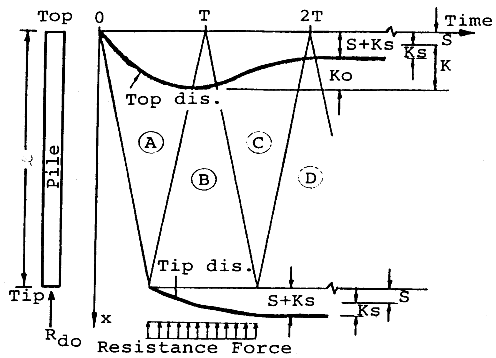

Figure 3 Wave Reflection Chart (after Uto et. al. (1985)

This solution is in the so-called "d'Alembert Form." From the solution of the wave equation can be conceptualised as an odd periodic function, the period being defined by the length of the vibrating rod. If this expansion is returned to the physical domain, it shows a series of wave reflections along the rod, as shown in Figure 3.

The method of images is based on this concept, because it seeks to solve the wave equation by considering the effects of the periodic transmissions and reflections of the stress wave generated by the hammer along the pile. In doing this it attempts to avoid the complexities of other closed form solutions.

Glanville et. al (1938): This study was one of the first comprehensive studies on stress waves in piles in general. It was occasioned by the problems encountered in the breakage of concrete piles during driving, both at the top and the toe. Equations (1) and (8) were used to develop equations to estimate the stress in the pile during driving, using the method of images. Because of the complexity of the equations, the results were reduced to a series of charts where a quantity of dimensionless stress was plotted against the ratio of hammer weight to pile weight. The charts could then be used to estimate pile stresses and resistance. The charts were applicable to concrete piles only, and this was and is a serious limitation to such solutions, because they were applicable to a limited universe of piles.

In addition to developing a solution to the wave equation, the authors continued Isaacs' (1931) work in addressing technical issues and experimental techniques that have enduring interest in pile dynamics. These include the instrumentation and data collection on stresses and forces in piles, including remote data gathering through "portable" equipment in a trailer, further research on the effect of the hammer cushion on the generation and effect of the pile stress wave (these were included in the analytical work,) drop tower testing on cushion material to determine the cushion stiffness, and further work on the relationship of ram weight to pile weight and cross section. An example of the data collected is shown in Figure 4, which shows recorded tension stresses in the midpoint of the pile during driving.

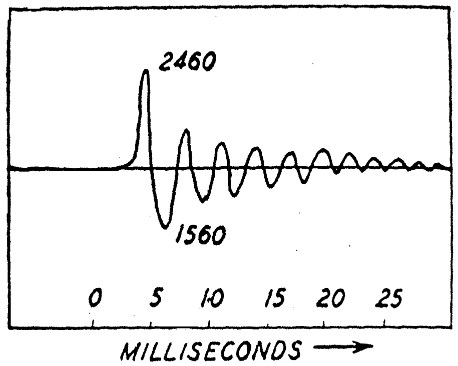

Figure 4 Recorded tension in the pile midpoint (in psi, after Glanville et. al. (1938))

Hansen and Denver (1980): The authors propose a solution of the wave equation using the method of images but also including a visco-elasto-plastic model for the shaft and toe friction. The behaviour of the pile is then calculated by successive applications of the stress wave. The authors also apply the model to pile discontinuities, both pile defects and changes in cross sectional area. The method is applied to a numerical integration technique for the analysis of actual piles.

Uto et. al. (1985): In this paper, a pile driving formula based on the solution of the wave equation is proposed. Neglecting shaft friction, toe damping, and making other assumptions concerning the displacement of the pile top and toe, the equation for the bearing capacity of the pile is given by the equation

$R_d = AE(S + K_s + 2K_0)/(2e_0) + NUL/e_f$..(9a)

or

$R_d = AEK/(Le_0) + NUL/e_f$..(9b)

where R_d = Pile bearing capacity, N

S = maximum displacement of the pile during driving, m

K = rebound of the pile during driving, m

K_s = rebound of the pile toe, m

K_o = rebound of the pile top, m

e_0 = correction factor for pile type

e_f = correction factor = generally 2.5

N = Average N value for the pile shaft

U = circumference of the pile shaft, m

Although these equations certainly use the method of images as a starting point, it is important to note that many "empirical" factors are taken into account to arrive at these formulas. The second term in each equation is not based on wave mechanics but Meyerhof's formulae for shaft friction. Also, since both maximum dynamic set and rebound are required, these equations are best applied in the field for verification of pile and hammer performance. Tada et. al. (1985) supply additional theory to arrive at Equation (9) and at the same time apply this equation to a hydraulic impact hammer, where they achieved good correlation in tests.

Fourier Series Solutions

Fourier series solutions are those which utilize an infinite series of orthogonal eigenfunctions to describe the motion and the stress on the pile. Equation (3) is such a solution. Fourier series are described in detail in Tolstov (1962). Included in these solutions are those which use Fourier integrals, Fourier transforms and inverse Fourier transforms. Following are descriptions of this type of solution:

Wang (1988): This study solved the wave equation directly using the method of weighted residuals. Using a plastic shaft resistance model, an elastic toe model, a uniform pile velocity at zero time and initial displacement, and no initial compression of the pile, the response was computed by the equation

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos(n\pi ct/L) + B_n \sin(n\pi ct/L)) \sin(n\pi x/L) + C_n x^2 + D_n x + E_n \qquad (10)$$

where A_n, B_n, C_n, D_n, E_n = constants based on integration by weighted residuals

Wang goes on to use this model not directly but as part of a finite difference scheme. This enables him to overcome the greatest weakness of the model, namely the assumption of a uniform pile velocity at impact, because in fact (assuming all velocity in the pile has gone to zero from the previous blow) only the particles at the pile top have any velocity at the time of impact.

Espinoza (1991): One of the characteristics that separates wave equation analysis in piles from analysis in other fields, such as acoustics and vibrations, is that in these fields the loads are considered as periodic (sine and cosine) functions. This simplifies the analysis and also enables the response to be considered spectrally rather than in actual movement. Except for vibratory pile driving, this has not been applied to pile driving analysis.

This study attempts to bridge the gap by first determining the displacement and force as a function of the spectral response of the system; the equations are

$$u(x,\omega) = P\,(e^{-ikx} + \lambda_t e^{-ik(2L-x)})/((1-\lambda_t e^{i2kL})ikEA) \qquad (11a)$$

and

$$F(x,\omega) = P\,(e^{-ikx} + \lambda_t e^{-ik(2L-x)})/(1-\lambda_t e^{i2kL}) \qquad (11b)$$

where F = force at a given point on the pile, N
P = force at the top of the pile, N (usually negative)
k = coefficient based on the pile, shaft soil damping and shaft soil elasticity
λ_t = coefficient based on the pile, toe soil damping and toe soil elasticity
ω = frequency, rad/sec

A Fourier transform was applied to the top of the pile to transform the hammer impact force into a spectrum of forces, and an inverse transform is necessary to obtain the force-time and displacement-time histories of the cases studied. The model was compared with finite difference and finite element techniques. The model was found to be most useful when pile displacements were small, because the soil was modelled visco-elastically without consideration of plasticity.

Solutions for Vibratory Hammers

Although the main point of interest here is with impact hammers, vibratory hammers have been the subject of serious investigation as well. The analysis of vibratory hammers has the advantage of dealing with a forcing frequency and the steady state solution. In this case the solution generally is in terms of only one frequency, which eliminates the infinite series. Although such a solution could be classified as a Fourier series of one term due to orthogonality, none of the authors cited below make this claim.

Hejazi (1963): This work is an extensive analysis of the theoretical aspects of vibratory pile driving. Part of this work consists of the derivation of equations of an elastic rod, penetrating the soil and subject to vibrations at the top. He divided the pile into two parts; the part penetrating the soil and the part which is above the soil; these are shown in Figure 5.

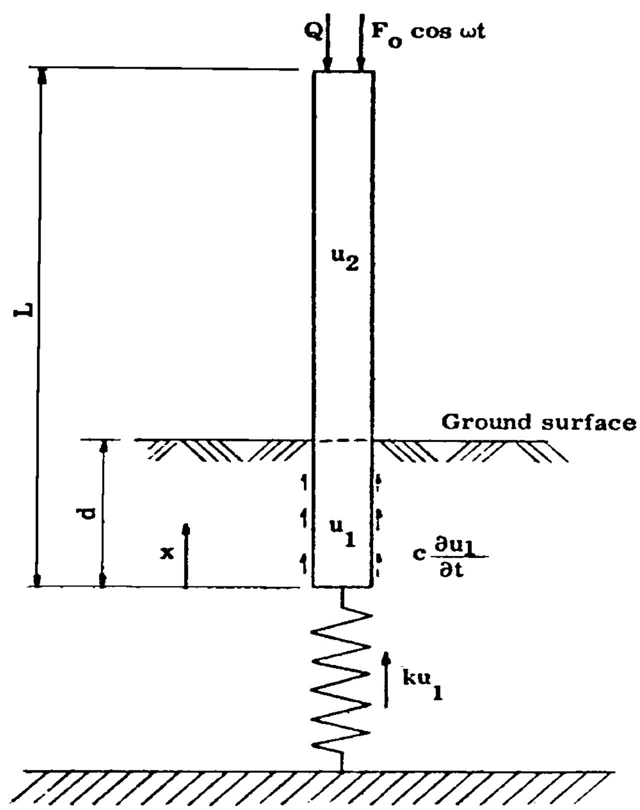

Figure 5 Diagram of Pile Model (after Hejazi (1963))

The steady state displacement for each of these parts is

$u_1(x,t) = (C_1 \cos(\alpha_1 x) + C_2 \sin(\alpha_1 x))\, e^{i\beta_1 t}$ .... (12)

$u_2(x,t) = (C_3 \cos(\alpha_2 x) + C_4 \sin(\alpha_2 x)) \cos(\beta_2 t)$ .... (13)

where $u_1(x,t)$ = displacement of the pile below the soil surface, m
$u_2(x,t)$ = displacement of the pile above the soil surface, m
$\alpha_1, \alpha_2, \beta_1, \beta_2, C_1, C_2, C_3, C_4$ = coefficients based on physical characteristics of system

The complexity of the solution led Hejazi to recommend using a rigid pile type solution for vibratory pile driving. Such a solution as this needed a computer to solve it expeditiously.

Smart (1970): In his analysis of vibratory piles, he proposes a model with a sinusoidal force at the top and viscous toe resistance (toe impedance) at the bottom. Using a d'Alembert type of solution, the force at any time is given by the equation

$F = F_0(Z_L\cos((\omega/c)(L-x)) + iZ_0\sin((\omega/c)(L-x)))\ /\ (Z_L\cos(\omega L/c) + iZ_0\sin(\omega L/c))$............(14)

where Z_0 = pile impedance (cross section is uniform), kg/sec
Z_L = soil impedance at toe, kg/sec
F_0 = amplitude of pile force at pile top, N

Semi-Infinite Pile Solutions

One special type of solution of the wave equation is the solution of the wave equation for semi-infinite piles. In these solutions, the pile is modelled as a semi-infinite bar, struck at one end. The result of this analysis is to model the interaction of the hammer and the pile only, generally without direct consideration of soil interaction or of numerical problems associated with hammer-pile modelling.

The central concept in semi-infinite pile modelling is mechanical impedance, which is defined as

$Z = F/V$ ....(15)

where Z = mechanical impedance of the pile, kg/sec
F = force for a given pile cross section, N
V = particle speed in the pile, m/sec

For a bar of uniform cross section, the mechanical impedance is given by the equation

$Z = \rho c A = \sqrt{(\rho E)}A$ ....(16)

where A = cross-sectional area of the pile, m²

If the impact of a solid ram on the end of the semi-infinite pile is considered, Equation (7) can be rewritten as

$\sigma = \sigma_0 e^{-(tZ/M)}$ .... (17)

and other hammer models can be similarly represented. Because there are neither other reflections nor the movement of the pile to consider, when this response is mapped to the distance domain along the pile, it represents the entire response of the system.

Such a system was suggested by Lowery et. al. (1969). The following are summaries of the various solutions to the problem:

Parola (1970): He first analysed the infinite pile model in a systematic way, formulating variables and computing them using an analogue computer. He also attempted to apply the results of the hammer-pile interaction at the top of the pile to the response of the soil. His work was confirmed and expanded by Warrington (1987) using numerical integration and included cushionless hammers as well as cushioned ones. He also attempted to apply the results of the hammer-pile interaction to soil response, using the method (itself a type of "closed form" solution of the wave equation) established by Kümmel (1984).

Van Koten et. al. (1980): They developed a semi-infinite pile solution which included visco-elastic shaft resistance; the resulting equation of displacement includes modified Bessel functions. The model is then converted to a finite pile model using the method of images.

Deeks (1993): This was a comprehensive solution of the equations of motion for the semi-infinite pile in true closed form with application to actual case histories. His main objective was to use these results to evaluate numerical methods of analysis for piles, an important application for closed form methods. Deeks also considered losses in the cushion material as viscous losses, which give the possibility of analysing variations in the loading rate of the cushion material, as opposed to the static one presently used with finite difference wave equation analyses.

Observations on Existing Closed Form Solutions

In analysing the future potential for closed form solutions to analyse stress waves in piles, some observations are in order on what has gone before.

- Most closed form solutions, especially those after Isaacs (1931) and Glanville et. al. (1938) are not really comprehensive, i.e., they are not intended to be used for the prediction of pile behaviour in its totality. They are designed to meet specific requirements. This is especially true for semi-infinite pile solutions, although efforts have been made to broaden these as well.
- The d'Alembert, method of images type of solutions are the most common but in most cases they do not consider the shaft friction. This is a serious omission considering the application of piles.
- Fourier type solutions are relatively rare because of the infinite nature of the equations but with the growth of computer capabilities they have more potential. In any case attempts to derive formulae and methods by other methods produced equations that in practical terms are little simpler than Fourier series.

- The constitutive modelling of the soils in closed form solutions has traditionally been rudimentary. It is very likely that, to accurately model the soil in these solutions, a different approach will have to be taken.
- Although the semi-infinite pile model seems to be a very special case, it is useful because it a) allows the analysis of a very important part of the system using relatively simple equations and without numerical difficulties, b) and it gives a simple method of computing pile and hammer loads and stresses for a wide variety of cases. It is interesting to note that much the work on this model after Parola (1970) has been primarily directed towards piles used in offshore platforms. Especially for piles driven from the surface of the water, the force-time profile generated at the top and the soil response at the bottom are essentially "decoupled" by the intervening length of the pile, and so the semi-infinite pile model has its best application where the piles are the longest.

Applications for Closed Form Solutions

Based on these and other considerations, there are several useful applications for these solutions; they are as follows:

1) Parametric Studies: Although finite difference programs can be used for parametric studies (Meseck, 1985,) most finite difference codes are not designed to be used parametrically, but on a "job to job" basis. Furthermore, any trends to be derived from these are either strictly qualitative in nature or reduced from numerical analysis, a technique more suitable for empirical data. Parametric studies are useful for such tasks as equipment design and general pile specifications.

2) Verification of Numerical Methods: In spite of the popularity of numerical methods, it has been shown that there are computational difficulties associated with them (van Weele and Kay, 1984; Davis and Phelan, 1988). Closed form solutions are a valuable tool in evaluation numerical methods for such difficulties, as illustrated by Deeks (1993).

3) Advances in Computer Software: One of the major reasons that closed form solutions were pursued in the first place was to reduce the computations involved for pile loads and stresses using stress-wave analysis to a manageable level. The complexities of the problem, however, have made that goal unrealised, even though the programming requirements of closed form solutions are not excessive by modern standards. With the advance of mathematical software, however, the potential exists of generating a solution to this problem in closed form without recourse to specialized software.

Requirements for Successful Closed Form Solutions

The rationale for the closed form solution having been described, our next step is to set for the requirements for successful closed form solutions in the future. The actual solution is beyond the scope of this paper; however, based on the research described, some important requirements for closed form solutions can be set forth.

In order to solve any ordinary or partial differential equation, three things must be defined: a) the differential equation itself, b) boundary conditions, and c) initial conditions.

Differential Equation: Although Equation (1) is cited in many papers on pile dynamics, it is essential to realize that for the most part it cannot be applied directly to piling. This is because, even though end conditions such as impact and toe resistance can be handled with the boundary conditions, shaft resistance must be included in the equation itself, even though it acts on a "boundary" of the system. A better starting equation for piling would be the transmission line or Telegrapher's equation, for piling given by

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) - (\mu_s/(\rho A))u_t(x,t) - (k_s/(\rho A))u(x,t) \quad (18)$$

where k_s = shaft soil spring constant, N/m^2
μ_s = shaft soil damping, $N\text{-sec}/m^2$

Although variations of this exist and are used, any worthwhile solution of the wave equation must take both shaft and toe resistance into consideration; however in using this equation two difficulties are encountered. One is that Equation (18) does not take into consideration any plastic deformation after passing the yield limit of the soil (Espinoza, 1991.) Since the plastic deformation of the soil is essential to the penetration of the pile, this will have to be considered. Another is that it assumes the properties of the soil to be constant along the length of the pile shaft.

Boundary Conditions: The shaft conditions having been dealt with in the differential equation itself, the boundaries that need to be dealt with are at the ends. At the top is the hammer, which has been modelled to varying degrees of accuracy. The pile toe has many of the same problems as the shaft, especially as they related to plastic deformation of the soil.

Initial Conditions: Most pile models start with both displacement and velocity at zero, and for the most part the closed form solutions can start with this as well. This points out an important difference between piling analysis and other wave equation problems, i.e. the excitation of the pile comes from the end of the pile rather than along its length in an initial condition.

Conclusion

Although they have not received the attention and investigation as their numerical counterparts have, closed form solutions offer a great deal of understanding of the characteristics of driven pile systems. They can serve both to perform special analysis types on piling and to enable correlation of numerical methods as well. Although it is unlikely that simple equations will result from closed form analysis, nevertheless with the analysis capabilities at hand this long standing obstacle may be overcome.

Acknowledgements

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Corps of Engineers, Waterways Experiment Station for their review and assistance in the preparation of this paper. Also, since this is a paper on foundational things, we end with a quotation from Moses Maimonides: "God is near all who call Him, if they call Him in truth, and turn to Him. He is found by every one who seeks Him, if he always goes towards Him, and never goes astray."

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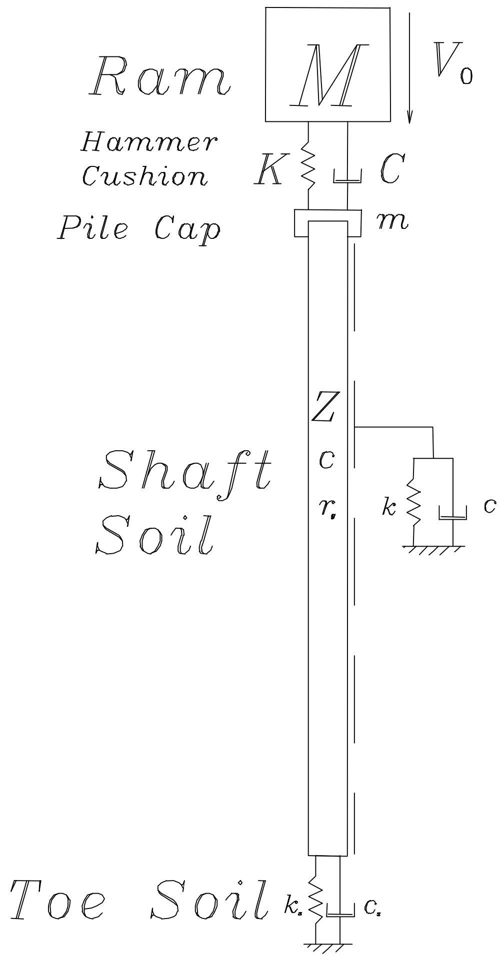

Now that x=L…

For those of us involved in the one-dimensional wave equation, this means that you have reached the end! We trust that the information presented in the article concerning the wave equation or other technical matters has been useful to you. We should now like to take the time to make some other observations.

It is our conviction that the beauty of our world and universe, especially as it is expressed mathematically but certainly in other ways, speaks of its formation by an intelligent Creator. This is underscored by the unity that appears both in mathematics and in the physical laws which mathematics are used in to quantify and qualify. As scientists and engineers we depend upon this unity to both make sense out of what we observe and to make progress both in our knowledge and in our application of that knowledge to practical problems.

But as we turn away from the reverie of beautiful formulations, we see a world that is marred by human failing. This manifests itself in many forms that we are reminded of daily. The longer we live on this earth the more those failings come home to inflict pain upon us, no matter how hard we try to escape them.

It was not God's intent to leave us with this pain alone in his creation but to offer us a way by which we finite beings be united into his perfect infinity, something which is both definable and beyond definition. In infinity past he was with his Son Jesus Christ and Jesus came to live amongst us, share our situation and ultimately face torture and execution by those who were threatened by his message.

But this was not the end, for Jesus being God rose from the dead and offers us both a way out of our present condition in this life and eternal life with God, not by simply following a set of rules but by having God himself live in us and both empowering and leading us in a better way. If we commit ourselves to Jesus then for us $L = \infty$, which means that we have life forever.

All of these things are described in the book called the *Bible*; but in the meanwhile you can learn more at the website

http://www.geocities.com/penlay

or by emailing us at uttc2uxx@geocities.com. We look forward to hearing from you.